FIRST YEAR CALCULUS

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RESPECTFULLY DEDICATED

TO

MY *GURU*

Rev. Fr. Louis Vion, S. J.

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Publisher's Note

As a book of scripture, the *Bhagavadgītā* has assumed a position of universal interest. Its teachings have gained appreciation not only in India, but far beyond its borders. Our Gītā-Library alone comprises about 1300 editions of the *Bhagavadgītā* published in 32 different languages including 8 foreign languages. And the Gita Press itself has struck off up to date 1,17,05,000 copies of this wonderful book within the last thirty-nine years. This is our apology for bringing out this English edition of the *Gītā* in pocket-size and in a popular form. We trust it will find

-avour with the English-reading public. The English translation of this edition has been based on the Hindi rendering of the *Gītā* made by Syt. Jayadayal Goyandka and appearing in the Gita-Tattva Number of the Hindi monthly 'Kalyan', published by the Gita Press. In preparing the present English translation, the translators have made use, every now and then, of other English translations of the *Gītā*, and express their grateful acknowledgement to the same.

In order to add to the utility of this small volume an introduction by Syt. Jayadayal Goyandka and a synopsis of the *Gītā* have been prefixed to the translation and an article by the same author bearing on the *Gītā* has been appended there to.

"The Great Architect of the Universe now begins to appear as a pure Mathematician".

Sir James Jeans.

PREFACE

This is a text-book covering the First Year Course in Calculus of the Poona University.

The prescribed course is concisely worded as follows:

> "Variables, functions, concept of a limit; derivatives of simple algebraic and trigonometric functions; derivatives of sums, products, quotients and functions of function; geometrical and physical interpretation of derivatives. Integration as the inverse process of differentiation."

It therefore leaves the teacher and the student free to read as much or as little as he likes between the lines.

Knowing this, I have arranged the matter in such a way that the teacher can pick and choose from any topic as much of it as he can conveniently deal with for his students to assimilate easily. Each topic is self-contained with its own set of a variety of illustrative examples and drill exercises that immediately follow, which an average student who closely follows the illustrative examples, should be able to solve without much difficulty. Having the 'Freshman' in view, I have purposely eschewed the idea of giving a miscellaneous set of examples at the end of each chapter or even at the end of the book.

When this newly introduced course in calculus for the first year class has been actually taught in the colleges for some time, then and then only, the experience of the teachers as well as that of the learners, will establish the exact scope and content of the syllabus.

To the student, however, I can say this:— "*This book has been specially written for you, having your needs and your capacity in view.*" I have given the explanations, particularly in the earlier part in great detail even at the risk of being lengthy, so that even the student who believes in self-help should find it easy to understand and to grasp the fundamentals of calculus.

It is difficult to hold the balance between mathematical rigour and simplicity. If there is a lapse of rigour anywhere, it was inevitable. For this lapse, I am sure the student, if not the teacher, will forgive me.

With reference to the treatment in this book vis a vis the syllabus, I have to admit that the temptation to deviate has been irresistible in certain parts.

I gratefully mention here the facility and encouragement given to me by Prof. K. V. Nayak of the Ruia College, Bombay while I wrote this book at Bombay. To Prof. D. B. Wagh of the Wilson College, Bombay I owe the perspective of presentation in this book. Sri. Achyut S. Apte, M. A., of the Hydrodynamic Research Station, Poona took immense pains to shape my press copy; I thank him too. I am obliged to Dr. D. K. Sen of the Rajaram College, Kolhapur for his having gladly permitted my publisher to publish this book on calculus in addition to the one by him published already.

I thank Prof. K. Balagangadharan of the Wadia College, Poona, Prof. Mrs. Madhumalati Apte of S. P. College, Poona and Prof. A. B. Shah of the Fergusson College, Poona for their valuable suggestions.

My thanks are also due to Sri. J. P. Apte, Art Master of our Society's High School at Deccan Gymkhana, Poona for the drawings in this book, and also to my brother, Yashwant who helped me in drawing the curves and in reading the proofs.

The management of the Aryabhushan Press also deserve my thanks for their careful execution of the printing of this volume.

All suggestions from the readers for the improvement of book will be gratefully received.

POONA 4.
July 12, 1950 }

M. N. BHAT.

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CHAPTER I

CONSTANTS AND VARIABLES

1·1. Nature of the subject.

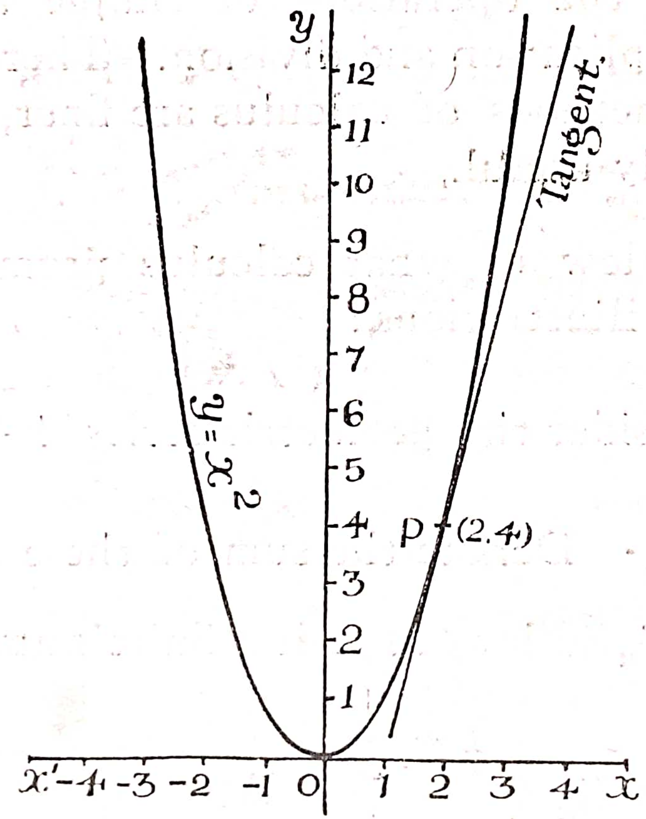
Calculus is a branch of Mathematics, and as the word suggests, calculus gives you some new methods of calculation, particularly needed to get results, not obtainable by the operations of simple addition, subtraction, multiplication and division. Therefore you will find that the methods of calculus are finer, powerful and therefore highly useful.

As a pre-view of what calculus promises to offer, here are a few illustrations :

(i) Consider the geometric series $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$. Denote the sum of these n terms by S_n. You can find S_n so long as n is a finite number. Perhaps you know that $S_n = \dfrac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}$. But just at present you are not sure of the sum of the series $1 + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ *ad infinitum* (i. e. unending). Calculus will give you this sum.

(ii) You know how to find the area of a triangle. Dividing any polygon (of any number of sides) you are able to find the area of a polygon too. But you do not know how to obtain the formula πr^2 for the area of a circle or again to establish the formula $\frac{4}{3}\pi r^3$ for the volume of a sphere and many other such common formulæ. Methods of calculus establish these.

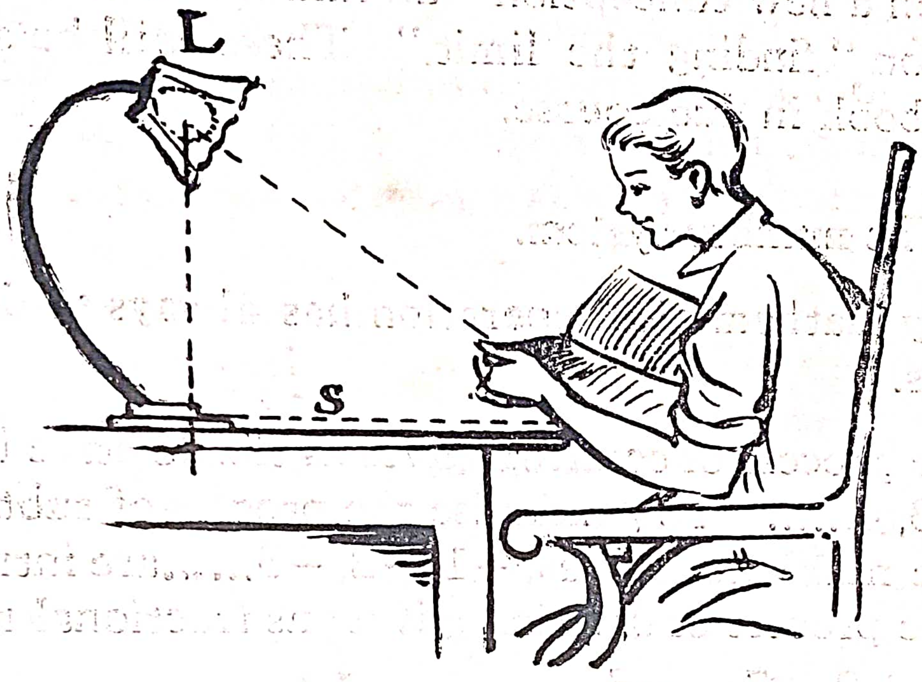
(iii) You know how to draw a tangent to a circle at any point on its circumference. The perpendicular at the point on the circumference to the radius drawn to that point will be tangent to the circle. But consider the graph of $y = x^2$. (2, 4) is a point on the graph. How to draw a tangent to the curve at (2, 4)?



Perhaps you will say, “ place the foot rule; adjust it near the point as best as you can and draw the line that touches but does not cut—as carefully as you can.” “—as carefully as you can ”—that is it, which means “ as accurately as you can.” Can you verify how accurate you could be or what was the error, if any? Well, you will be able to do this when you read chapter V of this book.

(iv) Physics tells you this :

The illumination of your small reading table is proportional to the cosine of the angle between the rays of light and the normal to the surface of the table and inversely proportional to the square of a distance of the lamp from the surface.



At what height should the lamp be placed in order that your book on the table *a* feet distant from the wall to which the lamp is fixed, may get the brightest illumination ?

Calculus alone offers a solution to the above problem.

In fact calculus offers a solution to many an intricate problem in almost every branch of science.

(v) Take the relation $y^2 = x^3$. The graph of this will evidently pass through the origin (0, 0). But mere graph will not give you more information about the nature of the curve very near the origin, but a study of $y^2 = x^3$ applying to it your knowledge of calculus will give you all the information about the curve at any point on it as if you have put it under a powerful microscope through which you are able to discern the minutest details of the object.

At this stage, I am sure, you are anxious to know the fundamental basis of this new subject of calculus whose methods are powerful and useful. Calculus is based on a new conception "an infinitesimal" and a new operation "finding the limit." These will be explained in this book in due course.

1·2. The number system.

Any mathematical operation has always to deal with numbers.

The process of counting gives us the positive integers, 0, 1, 2, 3, 4...... To generalise the process of subtraction, negative numbers such as, −1, −2, −3.......are introduced. Next the process of division gives us fractional numbers like $\frac{1}{3}, \frac{-2}{3}, \frac{7}{-8}, \frac{-3}{-4}$ etc.

Definition : A number $\frac{p}{q}$ where p and q are integers positive or negative (except that $q \neq 0$) is called a *rational number*.

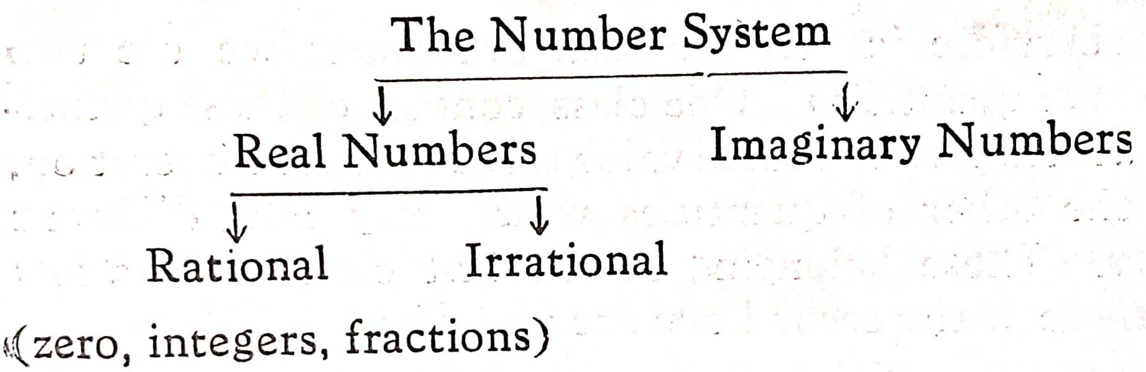
Integers are naturally included in the above definition, for an integer 5 can be looked upon as a fraction 5/1.

So the system of *rational* numbers consists of zero, integers (positive or negative) and fractions (positive or negative).

The class of numbers consists of other numbers besides rational numbers.

The numbers that we saw so far could always be expressed in the form $\frac{p}{q}$, where p and q are positive or negative integers ($q \neq 0$). There are, however, other numbers that cannot be expressed in the above form. $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{5}$, π are examples of such numbers. These numbers are called *irrational* numbers.

Numbers like $\sqrt{-2}$, $\sqrt{-4}$ etc. are *imaginary* numbers.



* In this book all the numbers involved are supposed to be real numbers.

1·3. The Fundamental Operations.

(A noteworthy exception)

We can *freely* do the four fundamental operations of addition, subtraction, multiplication and division with any real number, *except division by zero*.

In the case of *zero*, addition, subtraction and multiplication are defined, e. g. $x+0=x$, $0+0=0$, $x-0=x$, $0-0=0$, $x \times 0=0$, $0 \times 0=0$ etc. where x is any real number. But **division by zero is not defined**; $\frac{x}{0}$ where x is any real number, including zero, is meaningless.

* The student is advised to note this exception wherever it occurs.

1·4. Constants and variables.

The measure of a quantity is, in some cases, a constant and in other cases, it is a variable. The distance between Poona and Bombay is a constant. The speed of a train running between these two places is a variable. It varies not only at different stages of the journey of the same train, but also with different trains. The temperature of a place is a variable quantity whereas its altitude above sea-level is a constant.

Therefore in mathematical operations we use two classes of quantities. One class consists of those quantities which have the same value throughout the operation, and the other of quantities which may take different values. Those belonging to the first class are *constants* and those of the second one are *variables*.

Notation. Constants are generally denoted by the earlier letters of the alphabet, like a, b, c ...... k, l, m, n; the end letters s, t, u, v, w, x, y, z are used to denote variables.

For example, if s is the distance covered by a body moving with constant speed, in time t, then we say $s = kt$ where k is a constant to denote the constant speed. s and t are variables. s varies as t and vice versa. If we know the value of the constant k in the above relation, given the value of t, we can find out the corresponding value of s and vice versa.

Take another example, say, that of a hostel. The expenses of a hostel are partly constant, such as rent, establishment etc., and partly variable such as food charges etc. We can express this by the relation $y=ax+b$ where y stands for the total expenditure, x for the number of inmates, a for the mess-bill per head and b, the constant overhead expenditure. In this relation a and b are constants; x and y are variables. In fact the value of y depends on that of x. So we call x, the *independent variable*; y is called the *dependent variable.*

Interval of a variable. Sometimes a variable x can assume values lying between two fixed numbers only. In the above instance, if the hostel can accommodate only up to 40 students, then the relation $y = ax + b$ is true only for integral values of x between 0 and 40. We then say that x lies in the interval from 0 to 40, the interval being denoted by the symbol (0, 40).

The temperature of Poona on different days and at different hours of the day is a variable quantity. But we know that the lowest record did not go below 35°F and the highest one beyond 115°F. So we can say that the temperature of Poona varies in the interval (35°F, 115°F). Such an interval for a coastal city like Bombay will have a smaller range.

Let $y = \sqrt{4 - x}$. If y is known to be real, then when the expression $(4 - x)$ under the square root sign is negative, the square root cannot be extracted. So x here cannot be greater than 4.

Consider the expression $y = (x - 1)(x - 2)$ where, suppose it is given that y is always negative. Then it is easily seen that the range of the interval for the variable x is (1, 2).

Continuous real variable. A variable is said to be continuous in a given interval (a, b) if the variable can take for its value *any real number* contained in that interval. We deal mostly with continuous real variables.

Look at the graph of $y = x^2$ drawn elsewhere in this chapter, with which you are already familiar. To draw this graph, no doubt you plot only a few convenient points, but then you draw a *smooth curve* passing through all the points plotted. In $y = x^2$, given the value of x we can get the value of y and *also vice versa*; so you can consider either x or even y as the independent variable; then the other is the dependent variable. *Both these variables are continuous variables.* That is why you draw a smooth curve passing through the plotted points. From the graph, given any real value of x, you can find the corresponding value of y and *vice versa*.

1·5. Exercise.

1. In the equation $ax^2 + bx + c = 0$, which are the constants? If $a = 1$; $b = 2$ and $c = -3$, what are the values of x?

2. In the equation $ax^2 + 2hxy + by^2 = 0$, which are the constants? What are the variables? If $a=1$, $h = 2$, $b = 3$, express y in terms of x.

3. A railway engine without a train can go 30 miles an hour and its speed is diminished by a quantity which varies as the square root of the number of wagons attached. We express this as $s = 30 - k\sqrt{n}$. Which are the variables and which are the constants in this equation? When 16 wagons are attached, the speed of the train is 20 miles an hour. What is the range of interval of n here? Is n continuous?

4. $y = (1 + x)(2 - x)$ and y is positive. What is the range of interval for x? Is x continuous?

5. $y = \sqrt{9 - x^2}$. When y is real, what is the range of interval for x? Is x continuous?

6. $y = \dfrac{x-1}{x+1}$. For what value of x is y (i) equal to zero (ii) not defined (iii) equal to one (iv) equal to -1?

7. In the expressions below x and y are real. Find out their admissible values.

(i) $y^2 = (x-5)(x-2)$

(ii) $3y^2(1-x) = x(x-4)$ 9

(iii) $\sin^{-1} x$.

(iv) $\tan^{-1} x$.

(v) $y = x + \sqrt{x^2 - 1}$.

1·6. Idea of a function.

We have already referred to an independent variable and a dependent variable. We also illustrated how y depends for its value on the value assigned to x. In such a case y is said to be a *function* of x. x is called the *argument* of the function, e. g. if $y = 2x + 3$, then y is a function of x. x is the argument, 2 and 3 are constants.

Definition:—When two quantities are connected in such a way that to each value of one there corresponds a value of the other then each quantity is said to be a function of the other.

(i) In triangles on the same base, the area and the altitude are functions of each other.

(ii) The distance covered by a moving body and the time taken are functions of each other.

(iii) $x^2 + 2x - 6$ is a function of x.

Notation for functions:—Take the function x^2+2x-6. If $x=2$, then the value of the function is 2, and if $x=3$, then its value is 9. A compact way of denoting a function of x is by using the notation $f(x)$ where by $f(x)$ we here mean the expression x^2+2x-6. The two substitutions that we made can now be conveniently and in short form be indicated as follows:—

$$\begin{aligned} f(x) &= x^2+2x-6 \\ \therefore \quad f(2) &= 2^2+2\cdot2-6 \\ &= 4+4-6. \\ &= 2. \\ f(3) &= 3^2+2\cdot3-6 \\ &= 9+6-6 \\ &= 9. \\ \text{Similarly } f(1) &= 1^2+2\cdot1-6 \\ &= -3 \\ \text{and } f(0) &= 0^2+2\cdot0-6 \\ &= -6. \end{aligned}$$

Next consider the function $y=4x^2-5$. If we want to put this function in a short form, we cannot again use the same notation $f(x)$ because here the expression in x is different. We shall therefore put here that $y=\phi(x)$ where $\phi(x)$ stands for the expression $4x^2-5$. Then

$$\begin{aligned} \phi(2) &= 4\cdot2^2-5 \\ &= 11 \\ \phi(0) &= 4\cdot0^2-5 \\ &= -5 \\ \phi(-1) &= 4(-1)^2-5 \\ &= -1 \\ \phi(a) &= 4a^2-5 \\ &= 4a^2-5 \text{ and so on.} \end{aligned}$$

These examples show us that if we want to denote several functions in a short form, we need different notations for them. $f(x)$, $\phi(x)$, $F(x)$, $\psi(x)$ etc. are some of the functional notations commonly used. We have also seen that these notations are convenient to indicate substitutions for x. It is also clear that $f(x)$ does not mean f multiplied by x; x, however, must always be written within brackets.

Some types of functions.

The examples of functions that we gave so far were *algebraic*. They were *direct* or *explicit* functions. They are so called because in them y is given directly in terms of x. $y = 3x^2 - 4x + 5$ is another example of a direct or explicit function of x. Here y alone on one side is given to be equal to an expression in x on the other side. So it is easy to calculate the value of y for a given value of x. Another example of an explicit function is $y = \frac{3x+4}{x-2}$. Here when $x = 1$; $y = 7$. But for $x = 2$ the function is not defined. Why?

We have other kinds of functions too. For example the equation $3x^3 - 4x^2y + 7y^3 = 0$ Here y is determinate for a given value of x. y is a function of x. But x and y are mixed up in the expression; y is not put separately as equal to an expression in x. Such an expression is called an *implicit* function.

$y = \sin x$; $y = \cos x$ or $y = \sin x + \cos^2 x$ are examples of *trigonometric* functions. In these the values of x are either degrees or radians. In $y = \sin x + \cos^2 x$, if $x = 0$, then $y = 1$; if $x = \frac{\pi}{2}$, then $y = 1$ etc.

$y = \sqrt{x}$ is also an example of a function. In this function if y is real, then the function does not exist for negative values of x.

A function may be defined only for a certain interval of the argument. Take, for instance, the function $y = \sqrt{9 - x^2}$. We have square roots of positive quantities only. So $(9 - x^2)$ cannot be negative. That means x cannot be outside the interval $(-3, +3)$. The function is defined only for the interval $(-3, +3)$ of the argument x.

A function of a different nature is shown in the example below:

$$f(x) = 3x^2 - 5x \quad \text{when} \quad x < 1$$
$$f(x) = 0 \quad \text{when} \quad x = 1$$
$$f(x) = 2x + 1 \quad \text{when} \quad x > 1$$

Here $f(-1) = 3(-1)^2 - 5(-1)$

$= 8$

$f(0) = 3 \cdot 0^2 - 5 \cdot 0$

$= 0$

$f(1) = 0$ as defined

$f(2) = 2 \cdot 2 + 1$

$= 5.$ and so on.

Exercise: $f(x) = 1 - x$ when $x < -1$ or $x > 1$

$f(x) = 1 + x$ when $-1 \leqslant x \leqslant 1$

Find (i) $f(-1)$ (ii) $f(0)$ (iii) $f(1)$

(iv) $f(2)$ (v) $f(-2)$ (vi) $f(\cdot 5)$.

Ans.: (i) 0 (ii) 1 (iii) 2

(iv) -1 (v) 3 (vi) 1·5.

1·7. Illustrative examples.

1. If $f(x) = \frac{1}{x}$ show that

$$f(x+a) - f(x-a) = \frac{2a}{a^2 - x^2}$$

Solution : $f(x) = \frac{1}{x}$

$$\therefore \quad f(x+a) = \frac{1}{x+a}$$

Similarly $f(x-a) = \frac{1}{x-a}$

$$\therefore \quad f(x+a) - f(x-a) = \frac{1}{x+a} - \frac{1}{x-a}$$

$$= \frac{x-a-(x+a)}{x^2-a^2}$$

$$= \frac{-2a}{x^2-a^2} = \frac{2a}{a^2-x^2}.$$

2. If $f(x) = 1 - \cos 2x$, find $f(0°)$, $f(90°)$, $f(45° - x)$ and $f(45° + x)$.

Solution : $f(x) = 1 - \cos 2x$

$$\therefore \quad f(0°) = 1 - \cos 2 \times 0°$$

$$= 1 - \cos 0°$$

$$= 1 - 1$$

$$= \mathbf{0}.$$

$$f(90°) = 1 - \cos(2 \times 90°)$$

$$= 1 - \cos 180°$$

$$= 1 - (-1)$$

$$= \mathbf{2}.$$

$$\begin{aligned} f(45^\circ - x) &= 1 - \cos 2(45^\circ - x) \\ &= 1 - \cos(90^\circ - 2x) \\ &= \mathbf{1 - \sin 2x} \end{aligned}$$

$$\begin{aligned} f(45^\circ + x) &= 1 - \cos 2(45^\circ + x) \\ &= 1 - \cos(90^\circ + 2x) \\ &= \mathbf{1 + \sin 2x}. \end{aligned}$$

3. If $f(x) = a^x$ show that

$$\frac{f(x+1) - f(x-1)}{a^2 - 1} = f(x-1).$$

Solution:

$$\begin{aligned} \frac{f(x+1) - f(x-1)}{a^2 - 1} &= \frac{a^{x+1} - a^{x-1}}{a^2 - 1} \\ &= \frac{a^x\, a - \dfrac{a^x}{a}}{a^2 - 1} \\ &= \frac{a^x\left(a - \dfrac{1}{a}\right)}{a^3 - 1} \\ &= \frac{a^x(a^2 - 1)}{a(a^2 - 1)} \\ &= \frac{a^x}{a} \\ &= a^{x-1} \\ &= f(x-1). \end{aligned}$$

4. If $f(t) = t^3 + 6t^2 - 1$ and $g(t) = 3t^3 - 4t^2 + 7t$ for what values of t will $f(t) = g(t)$?

Solution: Putting $f(t) = g(t)$ we have

$$t^3 + 6t^2 - 1 = 3t^3 - 4t^2 + 7t$$

i. e. $2t^3 - 10t^2 + 7t + 1 = 0$

i. e. $(t-1)(2t^2-8t-1)=0$

$\therefore$ $t=1$ or $\dfrac{8\pm\sqrt{72}}{4}$ *i. e.* $2\pm\frac{3}{2}\sqrt{2}$

$\therefore$ $f(t)=g(t)$ when $t=1$ or $2\pm\frac{3}{2}\sqrt{2}$.

5. If $y=f(x)\equiv\dfrac{ax+b}{cx-a}$, prove that $x=f(y)$.

Solution: $f(x)=\dfrac{ax+b}{cx-a}$

$\therefore$ $f(y)=\dfrac{ay+b}{cy-a}$

We have $y=\dfrac{ax+b}{cx-a}$

$\therefore$ $cxv-ay=ax+b$

$x(cy-a)=ay+b$

$\therefore$ $x=\dfrac{ay+b}{cy-a}$

$\therefore$ $x=f(y)$.

1·8. Exercises on functional notation.

1. If $f(x)=\dfrac{x+2}{2x-3}$ find $f(1)$, $f(0)$, $f(x+2)$.

2. If $f(x)=x+\sin x$ [x being in radians]

find $f\left(\frac{\pi}{2}\right)$, $f(1)$ and $f(0)$.

3. If $f(x)=3x-4x^3$, find $f(0)$, $f(-2)$ and

$f\left(\dfrac{1}{1-x}\right)$.

4. If $f(x) = \dfrac{1}{\sqrt{x+1}}$ show that

$$f(x-1) - f(x) = \frac{1}{x\sqrt{x+1} + (x+1)\sqrt{x}}$$

5. Given that $f(x) = \dfrac{x-1}{x+1}$ show that

(i) $f\left(\dfrac{x-1}{x+1}\right) = -\dfrac{1}{x}$ (ii) $\dfrac{f(a)-f(b)}{1+f(a)\,f(b)} = \dfrac{a-b}{1+ab}$

6. If $f(x) = \dfrac{x-4}{4x-1}$ show that $f\{f(x)\} = x$.

7. If $f(x) = \dfrac{1+x}{1-x}$ show that $f(\tan\theta) = \tan\left(\dfrac{\pi}{4}+\theta\right)$

8. Given that $f(x) = a^x$ show that
$f(x) \times f(y) = f(x+y)$.

9. If $f(x) = \dfrac{1}{x}$ find the approximate value of

$\dfrac{f(3+h) - f(3)}{h}$ when h is very small.

10. Find in the following cases the approximate value of $\dfrac{\phi(x+h) - \phi(x)}{h}$ when h is very small compared to x.

(i) $\phi(x) = x^3 - 3$ (ii) $\phi(x) = \dfrac{1}{ax+b}$.

11. Given that $f(x) = \dfrac{2x-3}{x+1}$ and $\phi(x) = \dfrac{x+1}{2x-3}$
find $f\{\phi(x)\}$ and $\phi\{f(x)\}$.

12. If $f(x) = \dfrac{1}{x}$ and $\phi(x) = x$ show that
$f\{\phi(x)\} = \phi\{f(x)\}$.

13. Show that $\dfrac{2\cos^2 x - 1}{1 - 2\sin^2 x}$ is not a function of x.

14. If $f(n) = \sin^n \theta + \cos^n \theta$, show that

$$2f(6) - 3f(4) + 1 = 0.$$

1·9. Graphs of functions.

Given a function $y = f(x)$, we can draw its graph. The graph of a function, when drawn, presents to us vividly the nature of the function. Often the graph of a function enables us to study the function more clearly.

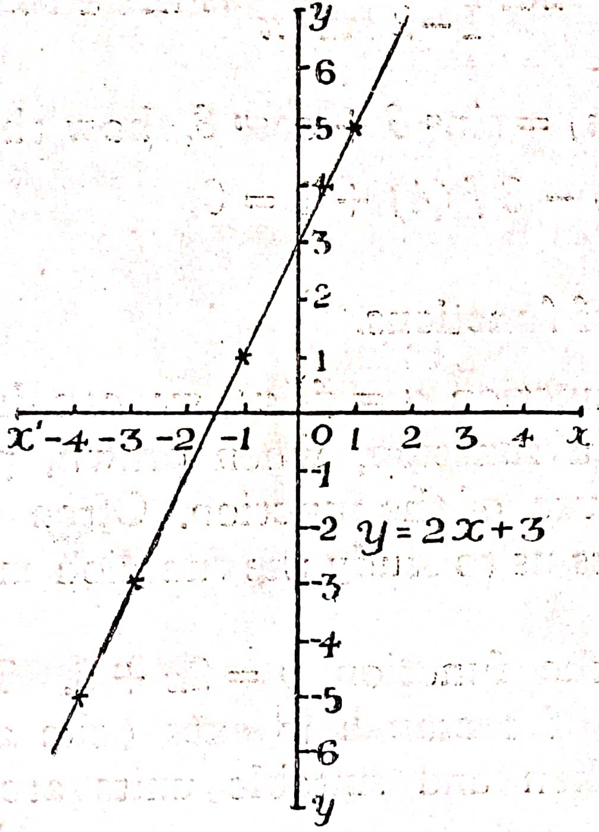
Consider the function $y = 2x + 3$. To draw its graph, a pair of rectangular axes (the x–axis and the y–axis) are taken and suitable units are marked on them.

The following table of corresponding values of x and y related to each other by the function $y = 2x + 3$, is then prepared.

$x =$	-4	-3	-2	-1	0	1	2	3
$y =$	-5	-3	-1	1	3	5	7	9

Plotting these points on the graph, we find that they lie in a st. line. This st. line is drawn. It is the graph of the function $y = 2x + 3$.

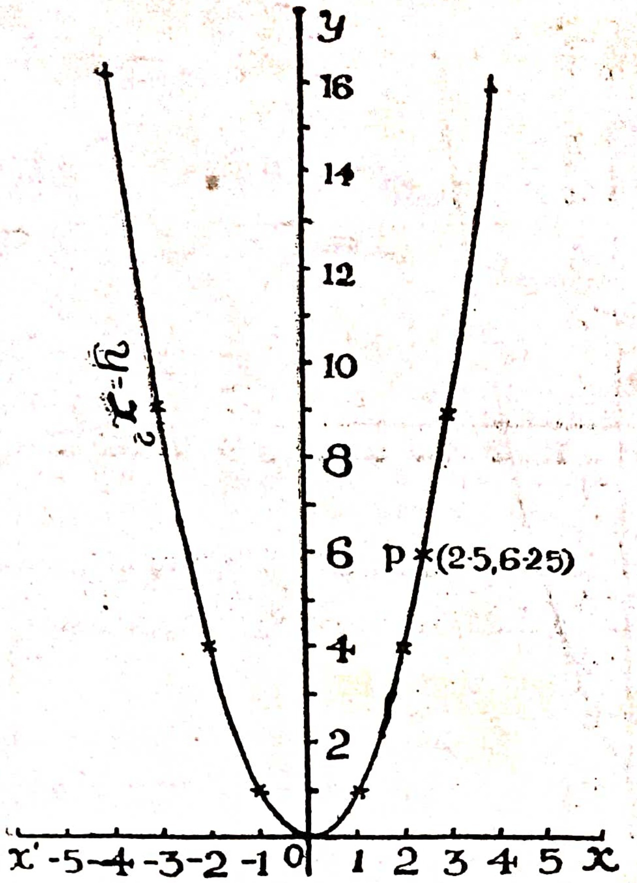
From the above graph we can read the value of y for any value of x. If necessary the line of the graph can be extended in either direction.



Below we give the table of values for the function $y = x^2$.

$x =$	-4	-3	-2	-1	0	1	2	3	4
$y =$	16	9	4	1	0	1	4	9	16

When the several points are plotted, we find that they are not in a st. line. However, they lie symmetrically about the y–axis. A smooth curve passing through these points should be drawn. That is the graph of the function $y = x^2$. This curve is called a parabola. The greater the number of points plotted in a range, the better will be the graph drawn.

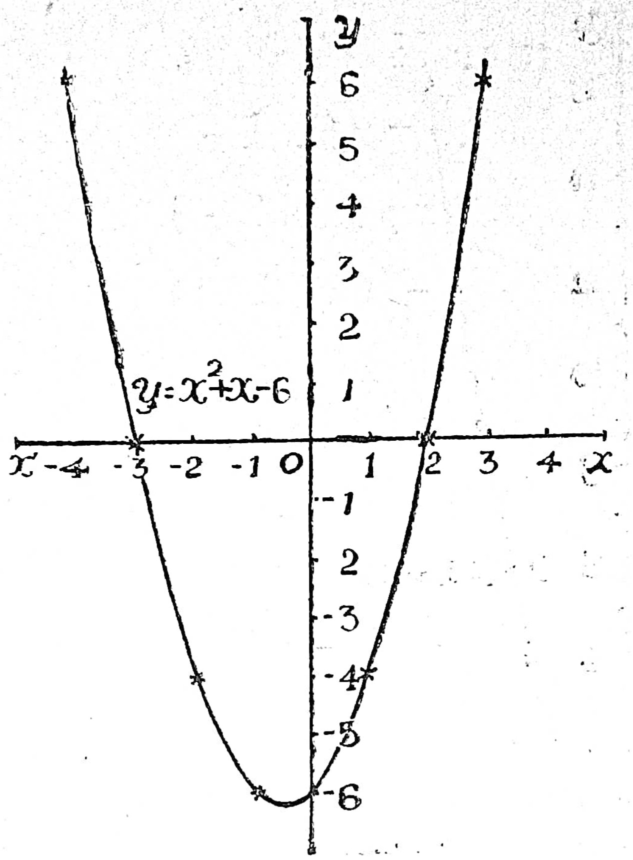


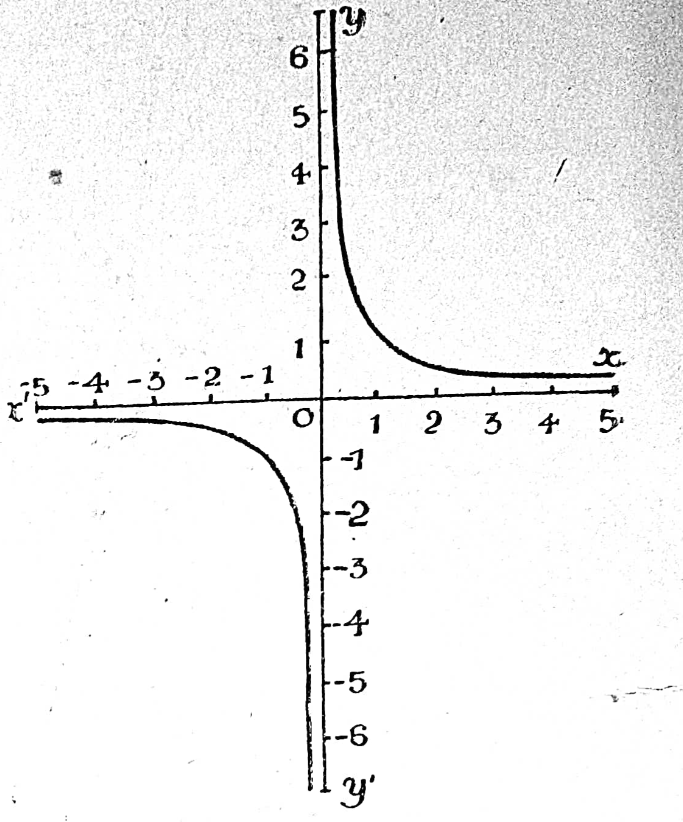
In the above graph draw the ordinate PP′ at $x = 2{\cdot}5$. It cuts the graph at P. The length of the ordinate PP′ represents 6·25 units. Looking at the function $y = x^2$ also we get when $x = 2{\cdot}5$; $y = 6{\cdot}25$ *i. e.* when the value of the argument x is 2·5, the corresponding value of the function y is equal to 6·25.

§ In general, when a graph of a function (usually called the *curve*) is drawn, the ordinate to the curve at any point x gives the corresponding value of y *i. e.* the value of the function.

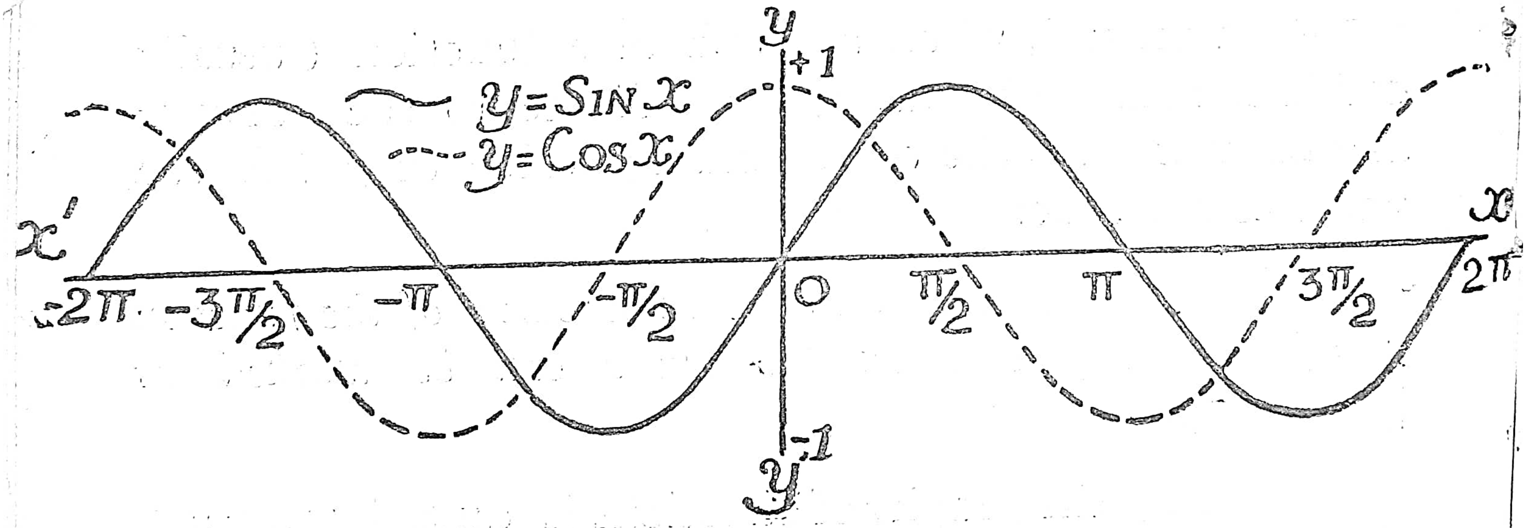
* It therefore follows that *in a curve of the function* $y = f(x)$, *any point on the curve can be marked as* $[x, f(x)]$.

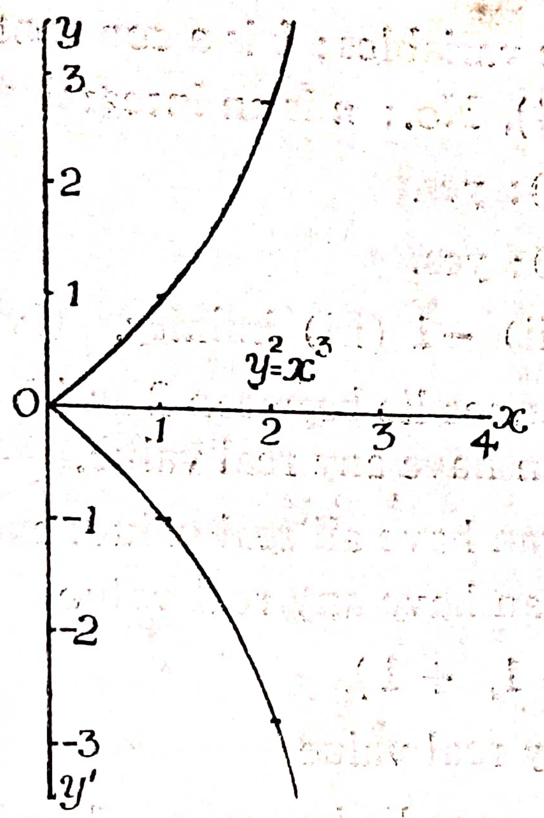
Drawing the graphs of functions is quite interesting, though not always easy. Here are some more simple curves. The student is advised to draw them accurately on graph paper, choosing suitable units.





The graph of $xy = 1$ or $y = \frac{1}{x}$





1·10. Exercises.

Draw the graphs of the following functions :—

(i) $x = 2$ (ii) $y = 3$ (iii) $y = x$ (iv) $x + y = 0$
(v) $y = 2x + 1$ (vi) $y = 2x^2$ (vii) $y = x^2 + x - 2$
(viii) $y = \sin x + \cos x$ (ix) $y = x^3$ (x) $y = 1/(x + 2)$.

Note :—When the graph of a function is to be drawn, if it is intended to study the nature of the curve drawn, then that curve should have its *true shape.* For drawing the curve with its true shape, it is *essential that the units taken on the x– and y– axes should be on the same scale.*

ANSWERS

Ex. 1·5.

1. a, b, c are constants. $x = 1$ or -3.

2. a, h, b are constants ; x, y are variables ;

$$y = -x \quad \text{or} \quad -\frac{x}{3}.$$

3. s, n are variables; k is a constant.
   (0, 144), No.; n is an integral variable.
4. $(-1, 2)$: yes.
5. $(-3, 3)$; yes.
6. (i) 1 (ii) -1 (iii) infinity. (iv) 0,
7. (i) x cannot lie between 2 and 5
   y can have any real value.
   (ii) x can have all real values except 1.
   y can have any real value.
   (iii) $(-1, +1)$.
   (iv) any real value.
   (v) x cannot lie between -1 and $+1$.
   y can have any real value.

Ex. 1·8.

1. $f(1) = -3;\ f(0) = -\frac{2}{3};\ f(x+2) = \dfrac{x+4}{2x+1}$.
2. $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + 1;\ f(1) = 1{\cdot}8415;\ f(0) = 0.$
3. $f(0) = 0;\ f(-2) = 26;\ f\left(\dfrac{1}{1-x}\right) = \dfrac{3x^2-6x-1}{(1-x)^3}$;

9. $\dfrac{-1}{9}$. 10. (i) $3x^2$ (ii) $\dfrac{-a}{(ax+b)^2}$.

CHAPTER II

LIMITS

2·1. Introduction.

In the study of functions, the usual algebraic processes have limited use. They do not, for example, help us to know if a function is an increasing function or decreasing one in an interval, what are its maximum or minimum values and so on. For this kind of study, calculus is more useful. As the basis of calculus, we introduce two new phrases, "tending to a limit" and "tending to infinity." We also formulate a new operation "finding the limit" which helps us to give a meaning to a function at a point where the direct substitution of the value of the argument is not possible.

We shall illustrate what we mean above by examples.

As already stated, "finding the limit" is the basic process in calculus.

2·2. The phrase "infinity" explained.

(i) Think of a large number. Perhaps you will say, 3481. Can you not think of a larger number? "Yes," you will say, and utter 1,84,753. But you can give out even a larger number than this. Is there any end to this process? Perhaps you will stop at a certain stage, say, millions of millions or billions of billions; but that is not the end. If you cannot think of larger numbers than these, another man might do so. This man again might be outdone by a third one. It is a matter of patience. But even patience might exhaust, but the number system does not. It is endless; *in other words*, it is "infinite." So if a number n grows larger and larger endlessly, we say that "n tends to infinity," by which we mean that n is capable of assuming values larger than you or I or anybody can think of. It follows that here "infinity" does not stand for any definite number.

(ii) Take the function $y = \frac{1}{x}$ and give to x successively smaller and smaller values. The corresponding values of y are larger and larger. If $x = 0{\cdot}001$, then $y = 1000$; if $x = 0{\cdot}000001$, then $y = 10{,}00{,}000$ and so on. This process is endless. This unending growth of the value of y as x becomes smaller and smaller is mathematically stated by saying that "$\frac{1}{x}$ *tends to infinity as* x *tends to* zero."

The last statement in the above paragraph needs further elucidation.

You can think of innumerable small values for x for substitution in $\frac{1}{x}$, each value nearer to zero than the others before; but the process of giving to x smaller and smaller values is *endless*. You cannot say, at any stage, "Now I have given you the smallest value. There is none smaller than this and nearer to zero than this." But the actual value zero to x is to be excluded, because then y becomes equal $\frac{1}{0}$ and division by zero is meaningless. Therefore we use the phrase, "tends to zero."

Conversely, at no stage will you be able to say, "Well, here is the largest possible value of y and none larger than this is possible." There is a limit to your capacity to conceive of larger and larger numbers, but there is no such limit on the expanding number system and when $y = \frac{1}{x}$, there is no limit to the large value that y may attain corresponding to a small value assigned to x. *This fact* is expressed by the phrase "the value of y *tends to* infinity." You will now agree with me when I say that it is very wrong or unmathematical to say "the value of y *is* infinity." For, from the explanation given above, you find that it is impossible to state the largest number and be sure that there is none larger than that. The phrase, "tending to infinity" just conveys this very fact.

Definition of "x tends to infinity"

Given a number N, however large, if a variable x can assume values greater than N, then x is said to tend to infinity.

This is written as $x \to \infty$. The arrow-mark stands for the phrase "tends to" and the symbol ∞ stands for the conception or idea "infinity".

There is no definite number "infinity"

It therefore follows, as "infinity" does not stand for any finite or definite number, you cannot add to, subtract from, multiply with or divide by "infinity" *in the ordinary way*. You strictly avoid such operations wherever the phrase "infinity" is involved, unless and until you know full well what meaning is attached to such operation with reference to their use with "infinity".

(iii) A professor once asked his students, "what is the number of stars in the sky?" And a boy answered "Infinity, Sir". Was he right? Definitely not! The literary statement. "The number of stars in the sky is infinite", may be tolerated because of our incapacity to count the actual number (and mind you, this number exists!). But mathematically that statement is wrong. "The number of stars in the sky *is infinity*" is also strictly tabooed; *it is meaningless.*

(iv) To fix up your idea of the new phrase, "tending to infinity", take the series $1 + 2 + 3 + 4 + \ldots + n$. Denote by S_n the sum of n such terms where n is variable. This sum grows larger and larger as n becomes larger and larger. We can therefore say that "S_n tends to infinity as n tends to infinity."

We put this as follows:—

$$\underset{n \to \infty}{S_n \to \infty}$$

Definition of a function tending to infinity as the variable tends to infinity.

The function $\phi(n)$ is said to tend to infinity as n tends to infinity, when any number n_0, however large, is given, we can find a number N such that for all values of $n > N$, $\phi(n) > n_0$

That is to say, however large n_0 may be, $\phi(n) > n_0$ for sufficiently large values of n.

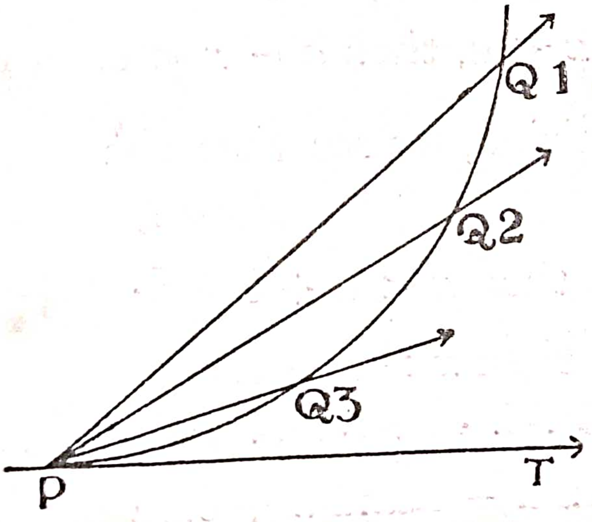
Less precisely we can say this, "$\phi(n)$ tends to infinity, if we can make $\phi(n)$ as large as we please by sufficiently increasing n."

2·3 Examples leading up to the idea of a limit.

(i) Suppose a regular polygon of n sides is inscribed in a given circle. Because the area depends upon the number of sides of the polygon, let the area of this polygon be denoted by $f(n)$. The greater the number of sides, the greater would be the area, but being an inscribed polygon, its area, in any case, cannot be greater than the area of the circle. At the same time, the greater the number of sides of the polygon, the more does the area of the inscribed polygon approximate to that of the circle. In other words, given a positive number ϵ (epsilon), however small, we can take n sufficiently large such that $f(n)$, the area of the polygon, differs from the area of the circle by less than ϵ. This fact is expressed by saying that the "limit of the area of the regular polygon of n sides inscribed in a circle as n tends to infinity, is the area of the circle."

In common language, we shall say that if the number of sides of the inscribed polygon is very very large, the difference between its area and the area of the circle is very, very small, almost negligible. What shall be this small negligible quantity? Well, anything *you* please to consider as negligible. So we call it ϵ, in general.

(ii) As another illustration of the idea of a limit, consider the *definition of a tangent* in geometry.



If P is a fixed point on a certain curve, and Q is another point on it near to P, we can draw the chord PQ.

As Q is taken nearer and nearer to P, we get different positions of the chord PQ. As Q is *indefinitely close to* P or in other words, as Q *tends to* P along the curve, the chord is called a tangent to the curve at P. Hence the tangent at P is the *limiting position* of the chord PQ as Q tends to P.

It is worth noting here that for the purpose of defining the tangent, it is *essential* that Q should be *distinct from* P, as two distinct points are necessary to define a st. line. Hence Q should not exactly coincide with P.

Meaning of $|x - a| < \delta$.

Before we proceed further, it is necessary to examine the meaning of the inequality $|x - a| < \delta$.

Now $|x - a|$ means the numerical (i. e. the positive) value of $x - a$, without regard to sign. For example $|7 - 3| = |3 - 7| = +4$. So $|x - a| < \delta$ means that the difference between x and a, taken positively, is less than δ, where of course, δ must be a positive number. This statement can be broken into two parts:

(i) if $x > a$, then $x - a < \delta$

(ii) if $x < a$, then $a - x < \delta$.

By transposition, we see that these are equivalent to the following:

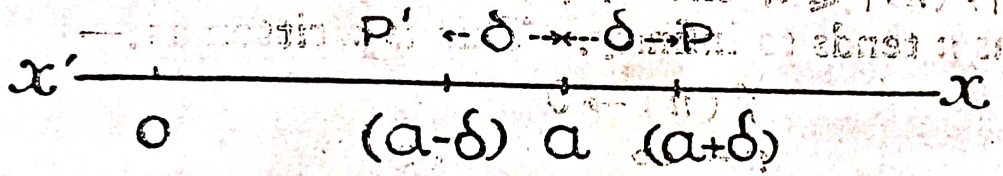
If $x > a$, then $x < a + \delta$.

and if $x < a$, then $x > a - \delta$.

i. e. whether x is greater than a or less than a, x must lie between $a - \delta$ and $a + \delta$.

Hence $|x - a| < \delta$ means that x can have any value between $a - \delta$ and $a + \delta$.

* The following figure is drawn to make the point vivid:



The point representing x can lie anywhere between P′ and P, i. e. the inequality $|x - a| < \delta$ defines a *neighbourhood* of $x = a$ which is of length δ on each side of it. (We may note, in passing, that the points P and P′ themselves are excluded).

Definition of "x tending to a".

Given a positive number δ, however small, if x can assume values such that $|x - a| < \delta$, then x is said to tend to a. This is written as $x \to a$.

A particular case of the above is when $a = 0$:

Given a positive number δ, however small, if x can assume values such that $|x| < \delta$ (i. e. the value of x taken numerically with its positive sign) then x is said to tend to zero. This is written as $x \to 0$.

As an illustration, let us take $x = \frac{1}{n}$. You know what is meant by n tending to infinity. As n becomes larger and larger, $\frac{1}{n}$ becomes smaller and smaller. As n tends to infinity, the value of $\frac{1}{n}$ approaches zero. So we can say here that x tends to zero.

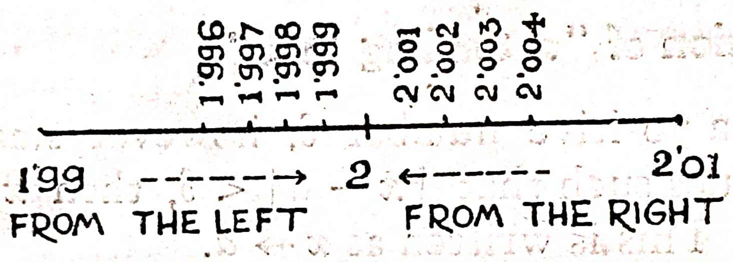
As x is a function of n, we can have the following definition too.

Given a positive number δ, however small, if we can find a number N, such that for all values of $n > N$ we have $|f(n)| \leqslant \delta$, then $f(n)$ is said to tend to the limit zero as n tends to infinity. This is written as:—

$$\underset{n \to \infty}{f(n) \to 0} \qquad \text{or}$$

$$\underset{n \to \infty}{\lim.}\, f(n) = 0$$

(iii) Take the function $y = x^2$. We shall study this function *in the neighbourhood* of $x = 2$ i. e. for values of n near enough to 2.



We have tabulated the values of y for different values of x, as x approaches 2 either from the left or from the right, as shown by the above diagram.

$x =$	1·99	1·997	1·998	1·999
$y =$	3·9601	3·9880	3·9920	3·9960

$x =$	2·01	2·003	2·002	2·001
$y =$	4·0401	4·0100	4·0080	4·0040

We note that as the value of x is taken nearer and nearer to 2, i. e. as the difference between x and 2 becomes smaller and smaller, the corresponding values of y go on approaching nearer and nearer to 4.

If you take more values of x which are nearer still to 2 (and there is no end to how near you can go to 2) the more convincingly you will realise that y approaches 4.

So we say that for the function $y = x^2$, y *tends to* 4 as x *tends to* 2 or that the *limit of* x^2 is 4 as x *tends to* 2.

To derive this conclusion we see that the values of x in the immediate neighbourhood of 2 are more important than those further away from 2.

We used above the phrase, "in the immediate neighbourhood of" which means that the interval of variation of x must be very very small. How small shall it be, on either side of 2? 'What is your idea of smallness? To me 0·01 was small enough. You might consider 0·001 more suitable, as it is smaller. Another might suggest even a still smaller figure. But the interval that we take should satisfy all; then only our conclusions are of any value. So it is better to use symbolic notation for the interval instead of taking concrete small numbers of our own choice.

Let us therefore use the symbol δ (delta) to denote *positive quantity, however small*. Give to x values lying between $(2 - \delta)$ and $(2 + \delta)$ only i. e. within the small interval $(2 - \delta,\ 2 + \delta)$. The corresponding values of the function i. e. of x^2 are $(4 - 4\delta + \delta^2)$ and $(4 + 4\delta + \delta^2)$ which shows that when x varies in the interval $(2 - \delta,\ 2 + \delta)$, y assumes only those values lying in the interval $(4 - 4\delta + \delta^2,\ 4 + 4\delta + \delta^2)$.

At this stage, for convenience let us introduce another symbol ϵ (epsilon) to denote $4\delta + \delta^2$. ϵ is greater than $(-4\delta+\delta^2)$. So we can now say that for the interval $(2-\delta, 2+\delta)$ of x, we have a corresponding interval $(4-\epsilon, 4+\epsilon)$ of x^2.

* Note that ϵ is a function of δ. It is equally true to say, conversely, that δ is a function of ϵ. The value of one depends upon the value of the other. Given the value of ϵ, we can get the corresponding value of δ by using the relation $\epsilon = 4\delta + \delta^2$ in the above case. Also, it is seen that the smaller the value of ϵ, the smaller the value of δ and vice versa. Like δ, ϵ is also positive.

We can now say that if $y = x^2$ then, given a positive quantity ϵ, however small, we can find a small positive quantity δ, such that for all values of x lying in the interval $(2-\delta, 2+\delta)$ the corresponding values of y lie within the interval $(4-\epsilon, 4+\epsilon)$.

Therefore we say that $\lim\limits_{x \to 2} x^2 = 4$.

In the above illustration we find that if $x = 2$, then $x^2 = 4$ exactly. The limit of the function as $x \to 2$ is the same as the actual value of the function when $x = 2$, So perhaps you might be templed to straightway substitute the given value of x in the function and put whatever value you get as the limit or the limiting value of the function. Such a process may not sometimes be possible; it is, strictly speaking, not correct to do so to find the limit, nor is it necessary. (We did not consider the case when $x = 2$ at all above).

The following example is meant to make the above statement clear.

(iv) Let $y = f(x) \equiv \dfrac{x^2-16}{x-4}$. We want to find the limit of the function as x tends to 4. Here we cannot substitute $x = 4$, because such a process gives us $y = \dfrac{16 - 16}{4 - 4} = \dfrac{0}{0}$ which is meaningless. In other words, *f (4) does not exist.*

So let us study the behaviour of the function in the neighbourhood of 4. For this purpose, choose an arbitrary small positive quantity δ and put $x = 4 + \delta$ in the above function. We have

$$\begin{aligned} y = f(x) &= \frac{(4 + \delta)^2 - 16}{(4 + \delta) - 4} \\ &= \frac{16 + 8\delta + \delta^2 - 16}{4 + \delta - 4} \\ &= 8 + \delta \\ &= 8 + \epsilon \text{ (here } \epsilon = \delta \text{)} \end{aligned}$$

As $\delta \to 0$, $\epsilon \to 0$ also.

Again, for values of x differing from 4 by less than δ, the corresponding values of $f(x)$ differ from 8 by less than ϵ.

Therefore we say that the limit of $f(x)$ as $x \to 4$ is 8.

If $y = f(x) \equiv \dfrac{x^2 - 16}{x - 4}$

$$\text{Then } \lim_{x \to 4} y \text{ i.e. } \lim_{x \to 4} f(x) = 8.$$

$$\text{or } \lim_{x \to 4} \left(\frac{x^2 - 16}{x - 4} \right) = 8.$$

* In the above example we learn that although the value of a function for a particular value of the variable does not exist, still the limit of the function as the variable tends to the particular value, may exist and if it exists, it can be found out.

* In finding the limit above, we never referred to $x = 4$ or $f(4)$ at all. We exclude the particular value $x = 4$.

See the following example:

(v) Evaluate $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x - 2}$.

Solution: $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x - 2} = \lim\limits_{x \to 2} \dfrac{(x-2)(x^2+2x+4)}{x - 2}$

$$\left.\begin{array}{l}\text{Dividing the numerator}\\ \text{and the denominator by}\\ (x-2) \text{ because } x \neq 2\end{array}\right\} = \lim_{x \to 2} (x^2 + 2x + 4)$$

$$= 4 + 2 \cdot 2 + 4$$

$$= 12.$$

* In the above working, we can legitimately divide the numerator and the denominator by $x - 2$ because $x \neq 2$. What we actually mean is that if δ is a positive quantity, however small, then $x = 2 + \delta$ or $2 - \delta$ and therefore $x \neq 2$. In the last step we take cognisance of the fact that δ ultimately tends to zero.

The formal definition of limit may now be stated.

Given a positive number ϵ, however small, if we can find a number δ, such that $|f(x) - l| < \epsilon$ for all values of x satisfying $|x - a| \leq \delta$, *except possibly* $x = a$, then $f(x)$ is said to approach the limit l as x tends to a.

This is written as $\lim\limits_{x \to a} f(x) = l$.

* Here a may be zero too. The student is advised to write out the definition for $\lim\limits_{x \to 0} f(x) = l$.

Find out $\lim_{x \to 0} \left(\frac{x^2 + 3x}{x}\right)$.

and show that it is equal to 3.

Let us give below another example in which the variable x tends to infinity. We shall find the limit of the function as $x \to \infty$.

(vi) Let $f(x) = \frac{x-1}{x+1}$

$$= \frac{x+1-2}{x+1}$$

$$= 1 - \frac{2}{x+1}.$$

We wish to examine this function as $x \to \infty$.

Take a very large value for x, say 99999. Then $x + 1 = 100{,}000$.

$$\therefore \quad \frac{2}{x+1} = 0{\cdot}00002$$

$$\therefore \quad f(x) = 1 - 0{\cdot}00002$$
$$= 0{\cdot}99998.$$

This shows that the larger the value of x that we take, the nearer will the value of $f(x)$ be to 1. For all values of x greater than 99999, the corresponding values of $f(x)$ will *always* differ from 1 by less than 0·00002. So we can say that as $x \to \infty$, $f(x) \to 1$.

But the large value of x is arbitrary. So denote it by the symbol N. Denote the *corresponding* small difference by ϵ. (When $N = 99999$, we saw that $\epsilon = 0{\cdot}00002$). Symbolically therefore ϵ is a function of N and vice versa.

Now we can say that given a positive number ϵ, however small, we can find a large number N such that for all values of x greater than N, the value of $f(x)$ differs from a number l by less than ϵ, then $f(x)$ tends to the limit l as x tends to infinity.

(vii) Let us take the geometric series

$$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots\dots + \frac{1}{2^{n-1}}$$

You know the formula for the sum in this case.

$$S_n = \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^n}\right)$$

As n is larger and larger, $\frac{1}{2^n}$ is smaller and smaller. In other words, $\frac{1}{2^n}$ tends to zero as n tends to infinity.

$$\therefore \quad \underset{n \to \infty}{S_n \to 2}.$$

This is also put as "$\lim\limits_{n \to \infty} S_n = 2$" or shortly as "$\underset{n \to \infty}{\text{Lt.}}\, S_n = 2$" or even as "$\underset{n \to \infty}{\text{L.}}\, S_n = 2$" and read as "limit of S_n (S – suffix – n) as n tends to infinity is equal to 2".

Definition of a function tending to a finite limit as the variable tends to infinity.

The function $f(n)$ is said to tend to the limit l as n tends to infinity if, given a positive number δ, however small, $f(n)$ differs from l by less than δ for sufficiently large values of n.

That is to say, given a positive number δ, however small, we can get a number N corresponding to δ, such that $f(n)$ differs from l by less than δ for all values of n greater than or equal to N.

Definition:—Given a positive number δ, however small, if we can find a corresponding number N such that $|f(n) - l| < \delta$ whenever $n \geqslant$, N, then $f(n) \rightarrow l$ as $n \rightarrow \infty$.

We write this as $\lim\limits_{n \rightarrow \infty} f(n) = l.$

* In the definition, n is an integral variable; so also in the illustrative example above. But we can also define, in a similar way, $\lim. f(x) = l$ where x is a continuous variable.

(iii) Lastly consider the function $\dfrac{x^2+4}{x-2}$. Here as $x \rightarrow 2$, the numerator tends to 8, whereas the denominator tends to zero. So as $x \rightarrow 2$, the function tends to infinity.

For example if $x = 1{\cdot}99999$, then

$$x - 2 = 0{\cdot}00001; \quad x^2 + 4 = 7{\cdot}9999599001.$$

$$\therefore \quad \frac{x^2+4}{x-2} = 799995{\cdot}99001.$$

Definition of a function tending to infinity as the argument tends to a finite value.

Given a positive number N, however large, if we can find a corresponding small positive number δ, such that for all values of x satisfying the condition $f(x) \geqslant N$ whenever $|x - a| < \delta$, then $f(x)$ is said to tend to infinity as x tends to a.

2·4 An important limit.

To show that $\lim\limits_{x \to 0} \left(\dfrac{\sin x}{x}\right) = 1,$

where x is measured in radians.

We shall estimate the value of $\lim\limits_{x \to 0} \left(\dfrac{\sin x}{x}\right)$ where x is measured in radians. We notice that the corresponding values of x and $\dfrac{\sin x}{x}$ are as follows:—

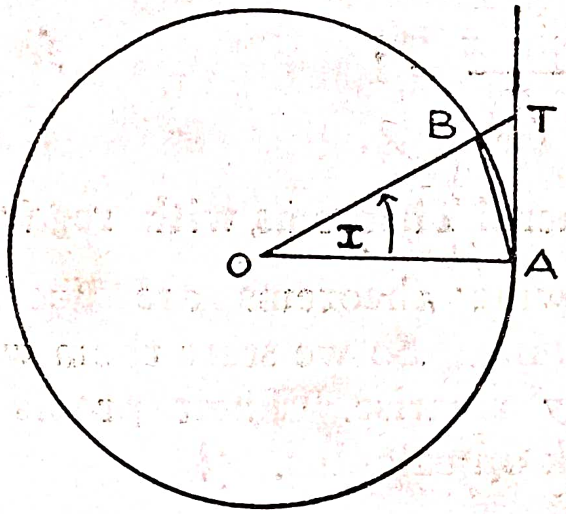
x	$\left(\dfrac{\sin x}{x}\right)$	x	$\left(\dfrac{\sin x}{x}\right)$
0·5	0·9589	−0·5	0·9589
0·3	0·9851	−0·3	0·9851
0·1	0·9983	−0·1	0·9983
0·05	0·9996	−0·05	0·9996
0·01	0·999998	−0·01	0·999998

It is apparent that $\left|\dfrac{\sin x}{x} - 1\right|$ will be as small as we please for all values of x for which $|x - 0|$ i. e. $|x|$ is sufficiently small. In common language, we can say, that for very small numerical value of x, measured in radians, $\sin x = x$ nearly.

We can therefore infer that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

The above limit can be formally established as follows:—



In a circle of radius r, let two radii OA and OB contain a small angle x $\left(< \frac{\pi}{2}\right)$. Let the tangent to the circle at A meet OB produced in T. Join AB also. Now area of $\triangle$ OAB < area of sector OAB < area of $\triangle$ OAT

$$\therefore \quad \tfrac{1}{2}\,OA \cdot OB \sin x < \tfrac{1}{2} r^2 x < \tfrac{1}{2}\,OA \cdot AT$$

$$\text{i. e.} \quad \tfrac{1}{2} r^2 \sin x < \tfrac{1}{2} r^2 x < \tfrac{1}{2} r \cdot r \tan x$$

(Here note that the area of the sector is $\frac{1}{2} r^2 x$ where x is measured in radians).

Divide the above inequality throughout by $\frac{1}{2} r^2$.

$$\therefore \quad \text{Sin } x < x < \tan x$$

Divide by sin x which is positive for a small value of x,

$$\therefore \quad 1 < \frac{x}{\sin x} < \frac{\cancel{x}1}{\cos x}.$$

Take the reciprocals of each of the three positive quantities,

$$\therefore \quad 1 > \frac{\sin x}{x} > \cos x$$

This shows that $\frac{\sin x}{x}$ always lies between 1 and $\cos x$. But $\cos x \to 1$ as $x \to 0$.

$$\therefore \quad \lim_{x \to 0} \frac{\sin x}{x} = 1$$

2·5 Some general Theorems with regard to limits.

The following theorems are necessary to solve examples on limits. So we state them without proofs in this elementary treatise. Their proofs can be found in any higher book on calculus.

Theorem I If $f(x)$ and $\phi(x)$ tend to limits a and b, then $f(x) \pm \phi(x)$ tends to the limit $a \pm b$.

Theorem II If $\lim. f(x) = a$ and $\lim. \phi(x) = b$, then $\lim. f(x) \times \phi(x) = ab$.

Theorem III If $\lim. f(x) = a$ and c is any constant, then $\lim. c \times f(x) = ca$.

Theorem IV If $\lim. f(x) = a$ and a is not zero, then $\lim. \left(\frac{1}{f(x)}\right) = \frac{1}{a}$

Theorem V If $\lim. f(x) = a$ and $\lim. \phi(x) = b$ and b is not zero, then

$$\lim. \frac{f(x)}{\phi(x)} = \frac{a}{b}.$$

2·6 Illustrative Examples:

1. Evaluate $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}$.

Solution:— $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x-1)}{x - 2}$

$$(\because x \neq 2) \quad = \lim_{x\to 2} (x-1)$$
$$= 2 - 1$$
$$= 1$$

Note :—In the above type of example, let us see what happens if $x - 2$ is not a factor in the numerator.

Suppose we want $\lim\limits_{x\to 2} \dfrac{x^2+3x+2}{x-2}$.

we have $\dfrac{x^2+3x+2}{x-2} = \dfrac{x(x-2)+5(x-2)+12}{x-2}$

$$= x+5+\frac{12}{x-2}.$$

As $x \to 2$, $\dfrac{12}{x-2} \to \infty$.

So $\dfrac{x^2+3x+2}{x-2}$ does not tend to a finite limit as $x \to 2$. It tends to infinity.

2. Evaluate $\lim\limits_{x\to 0} \dfrac{2x^2+5x-7}{x^3-3x+7}$.

As $x \to 0$, the numerator tends to -7 and the denominator tends to $+7$.

$\therefore$ The limit as $x \to 0 = \dfrac{-7}{+7} = -1$

3. Evaluate $\lim\limits_{n\to\infty} \left[1+\dfrac{(-1)^n}{n}\right]$

As $n \to \infty$, $(-1)^{n}$ oscillates between -1 and $+1$ according as n is odd or even. $\therefore$ $\dfrac{(-1)^n}{n} \to 0$ as $n \to \infty$.

$\therefore$ The limit of $\left[1+\dfrac{(-1)^n}{n}\right]$ as $n \to \infty$ is equal to 1.

4. Show that $\lim\limits_{x \to \infty} \left(\sqrt{x+1}-\sqrt{x}\right)=0.$

Solution :— $\lim\limits_{x \to \infty} \left[\sqrt{x+1}-\sqrt{x}\right]$

$$= \lim_{x \to \infty} \frac{\left[\sqrt{x+1}-\sqrt{x}\right]\left[\sqrt{x+1}+\sqrt{x}\right]}{\sqrt{x+1}+\sqrt{x}}$$

$$= \lim_{x \to \infty} \frac{x+1-x}{\sqrt{x+1}+\sqrt{x}}$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{x+1}+\sqrt{x}}$$

$= 0$ because as $x \to \infty$, $\sqrt{x+1}$ and $\sqrt{x} \to \infty$ also.

5. Evaluate $\lim\limits_{x \to \infty} \left[\dfrac{1}{n^3}\sum_1^n r^2\right]$

We know that $\sum\limits_1^n r^2 = \dfrac{1}{6} n(n+1)(2n+1)$

$$\therefore \operatorname{Lim.}_{n \to \infty} \left[\frac{1}{n^3}\sum_1^n r^2\right] = \lim_{n \to \infty} \left[\frac{1}{6n^3} n(n+1)(2n+1)\right]$$

$$= \lim_{n \to \infty} \frac{(n+1)(2n+1)}{6n^2}$$

$$= \lim_{n \to \infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$= \frac{1}{6} \times 1 \times 2 = \frac{1}{3} \text{ because}$$

as $n \to \infty, \dfrac{1}{n} \to 0.$

6. Show that $\lim\limits_{x \to \infty} \dfrac{3x^2 - 4x + 5}{2x^2 + x - 1} = \dfrac{2}{3}$.

Solution :—Put $y = \dfrac{1}{x}$ so that

as $x \to \infty$, $y \to 0$.

$$\therefore \lim_{x \to \infty} \frac{3x^2 - 4x + 5}{2x^2 + x - 1} = \lim_{y \to 0} \frac{\dfrac{3}{y^2} - \dfrac{4}{y} + 5}{\dfrac{2}{y^2} + \dfrac{1}{y} - 1}$$

$$= \lim_{y \to 0} \frac{3 - 4y + 5y^2}{2 + y - y^2}$$

$$= \frac{3}{2}.$$

7. Show that $\lim\limits_{x \to \frac{\pi}{2}} (\sec x - \tan x) = 0$

Solution :—Put $x = \dfrac{\pi}{2} - y$ so that

as $x \to = \dfrac{\pi}{2}$, $y \to 0$.

$$\therefore \lim_{x \to \frac{\pi}{2}} (\sec x - \tan x) = \lim_{x \to 0} \frac{1 - \sin x}{\cos x}$$

$$= \lim_{y \to 0} \frac{1 - \cos y}{\sin y}$$

$$= \lim_{y \to 0} \frac{2 \sin^2 \dfrac{y}{2}}{2 \sin \dfrac{y}{2} \cos \dfrac{y}{2}}$$

$$\left(\text{as } \sin\frac{y}{2} \neq 0\right) \quad = \lim_{y \to 0} \tan\frac{y}{2}$$

$$= 0$$

8. Show that $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2} = \dfrac{1}{2}$.

Solution :— $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2} = \lim\limits_{x \to 0} \dfrac{2 \sin^2 \frac{x}{2}}{x^2}$

$$= \lim_{x \to 0} \frac{\sin^2 \frac{x}{2}}{\frac{x^2}{2}}$$

$$= \lim_{x \to 0} \frac{\sin^2 \frac{x}{2}}{2 \cdot \frac{x^2}{4}}$$

$$= \lim_{x \to 0} \frac{1}{2}\left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2$$

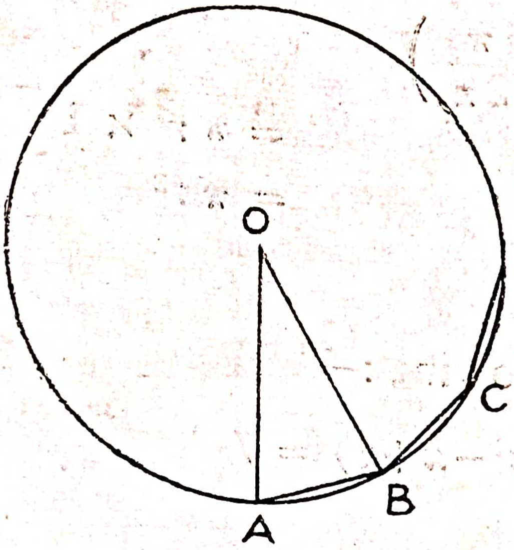
$$= \frac{1}{2} \lim_{x \to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2$$

$$= \frac{1}{2} \times 1$$

$$= \frac{1}{2}.$$

9. If a regular polygon of n sides is inscribed in a circle, the area of this polygon tends to be equal to the

area of the circle as $n \to \infty$. Using this obtain the formula πr^2 for the area of the circle.



$$\text{Area of the polygon} = n \times \text{area of } \triangle OAB$$

$$= n \times \tfrac{1}{2}\, OA \cdot OB \cdot \sin A\hat{O}B$$

$$= \frac{n}{2} r^2 \sin \frac{2\pi}{n}.$$

$$\therefore \quad \text{Area of the circle} = \lim_{n\to\infty} \frac{nr^2}{2} \sin \frac{2\pi}{n}$$

$$= \frac{r^2}{2} \lim_{n\to\infty} \left(n \sin \frac{2\pi}{n} \right)$$

$$= \frac{r^2}{2} \lim_{n\to\infty} \frac{\sin \frac{2\pi}{n}}{\frac{1}{n}}$$

$$= \frac{r^2}{2} \lim_{n\to\infty} 2\pi \left(\frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} \right)$$

$$= \frac{2\pi r^2}{2} \lim_{n\to\infty} \left(\frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} \right)$$

$$= \pi r^2 \lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta} \right)$$

$$\left(\text{where } \theta = \frac{\pi}{2n} \right)$$

$$= \pi r^2 \times 1$$

$$= \pi r^2$$

2·7 Exercises.

Prove the following :—

1. $\lim_{x \to -2} (x^2 - x + 1) = 7$

2. $\lim_{x \to \infty} \dfrac{x^2 - 3x + 2}{2x^2 - 4x + 3} = \dfrac{1}{2}$

3. $\lim_{x \to 1} (3x^3 - 7x + 4) = 0$

4. $\lim_{x \to 3} \dfrac{x^2 - 4x + 3}{2x^2 - 11x + 15} = 2$

5. $\lim_{x \to 2} \dfrac{\sqrt{3 - x} - 1}{2 - x} = \dfrac{1}{2}$

6. $\lim_{x \to \infty} \dfrac{2x^2 - 3x + 4}{x^2 - x + 5} = 2$

7. $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1} = 2$

8. $\lim_{x \to 0} \dfrac{\sin x^\circ}{x^\circ} = \dfrac{\pi}{180}$

9. $\lim_{x \to \infty} \dfrac{3x^3 - 5x + 1}{2x^3 - 4x^2 + 3x - 2} = \dfrac{3}{2}$

10. $\lim_{x \to \infty} \dfrac{6 - x^2}{x^2 + x - 2} = -1$

11. $$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{1}{2\sqrt{x}}$$

12. If $f(x) = 3x + x^2$, then $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 3 + 2x.$$

13. $$\lim_{x \to 1} \frac{x^2 - 1}{x^2 - 4x + 3} = -1$$

14. $$\lim_{x \to 2} \frac{x - 2}{x^2 + x - 6} = \frac{1}{5}$$

15. $$\lim_{x \to 0} x \sin \frac{1}{x} = 0$$

16. $$\lim_{x \to 3} \left(x^3 - 3x^2 + 4 - \frac{3}{x} \right) = 3$$

17. $$\lim_{x \to 2} \frac{x^2 - 7x + 10}{x + 5} = 0$$

18. $$\lim_{x \to 2} \frac{x^2 - 5}{x - 2} = \infty$$

19. $$\lim_{x \to 1} \frac{x^5 - 1}{x^7 - 1} = \frac{5}{7} \quad \left[\text{Hint } \frac{x^5 - 1}{x^7 - 1} = \frac{(x^5 - 1)/(x - 1)}{(x^7 - 1)/(x - 1)} \text{ etc.} \right]$$

20. $$\lim_{x \to 4} \frac{x^2 - 4x}{x^2 - 6x + 8} = 2$$

21. $$\lim_{x \to 1} \frac{x^4 - 2x^3 + 2x - 1}{x^4 - 2x^3 + 2x^2 - 2x + 1} = 0$$

22. $$\lim_{x \to 0} \frac{\sqrt{1 + x} - \sqrt{1 - x}}{x} = 1$$

23. $$\lim_{x \to 1} \left\{ \frac{1}{x-1} - \frac{1}{x^2 - x} \right\} = 1$$

24. $$\lim_{x \to 0} \frac{\sin 2x}{x} = 2$$

25. $$\lim_{x \to 0} \frac{\sin 5x}{\sin 2x} = \frac{5}{2} \left[\text{Hint } \frac{\sin 5x}{\sin 2x} = \frac{\dfrac{\sin 5x}{5x}}{\dfrac{\sin 2x}{2x}} \cdot \frac{5}{2} \text{ etc.} \right]$$

26. $$\lim_{x \to 0} \frac{x}{\sin^{-1} x} = 1$$

27. $$\lim_{x \to 0} \frac{1 - \cos x^2}{x^4} = \frac{1}{2}$$

28. $$\lim_{\theta \to 0} \frac{1 - \cos m\theta}{1 - \cos n\theta} = \frac{m^2}{n^2}$$

29. $$\lim_{x \to 0} \frac{\tan x - \sin x}{x^3} = \frac{1}{2}$$

30. $$\lim_{x \to 0} \frac{ax^2 + bx + c}{a'x^2 + b'x + c'} = \frac{c}{c'}$$

31. $$\lim_{x \to \infty} \frac{ax^2 + bx + c}{a'x^2 + b'x + c'} = \frac{a}{a'}$$

32. $$\lim_{x \to \infty} \frac{x^2 - 1}{\sqrt{x^4 + 1}} = 1$$

33. $$\lim_{x \to a} \frac{x^{\frac{1}{3}} - a^{\frac{1}{3}}}{x^{\frac{1}{9}} - a^{\frac{1}{9}}} = 6\, a^{\frac{5}{9}}$$

34. $$\lim_{x \to \infty} \frac{\sin x}{x} = 0$$

35. $\lim\limits_{x \to \frac{3\pi}{2}} (\sec x + \tan x) = 0$

36. $\lim\limits_{y \to 4} \left\{ \dfrac{\sqrt{y-4}}{\sqrt{y-3}-1} \right\} = \infty$

37. $\lim\limits_{x \to \infty} \{ \sqrt{x}\,(\sqrt{x+a} - \sqrt{x}) \} = \dfrac{a}{2}$

38. $\lim\limits_{x \to \infty} 2nr \sin \dfrac{\pi}{n} = 2\pi r.$ Use this to prove the formula for the length of the circumference of a circle.

39. $1{\cdot}\dot{2} = 1{\cdot}2222\cdots = 1\tfrac{2}{9}$. Prove it using the idea of a limit.

40. $\lim\limits_{n \to \infty} \dfrac{(-1)^n}{n} = 0.$

CHAPTER III

THE DERIVATIVE

3·1 The difference notation.

(i) Suppose the velocity of a moving body at a particular instant is 3 ft./sec. and at the next instant it is observed to be 4 ft./sec. The difference between the two velocities is 1 ft./sec. The velocity has been increasing. We can indicate this fact also by putting the difference as +1 ft./sec. If the velocities at any instant and at a *subsequent* instant had been 8 ft./sec. and 5 ft./sec. respectively, i. e. had the velocity been decreasing, we would put the difference in velocities as −3 ft./sec. The positive or negative sign attached to the difference gives us more information about the nature of the variation of the velocity, *as the time progresses.* " As the time progresses " is important here for our investigation of the variation of the velocity of the body.

Symbolically if t_0 and t_1 are two instants at which the velocities are respectively v_0 and v_1, then $t_1 - t_0$ is the interval of time during which the change in velocity is $v_1 - v_0$, where $t_1 - t_0$ is positive whereas $v_1 - v_0$ may be positive or negative.

The letters δ (delta – small letter) or Δ (delta – capital letter) of the Greek alphabets, corresponding *d* or D of the English alphabet are used conveniently to show the difference. This choice of the symbol is very appropriate, *d* – being the first letter of the word " difference," But the mere letter δ or Δ cannot tell us the quantities whose difference is taken. To give this information also, we use the notation δt or Δt to denote difference in time and δv or Δv to denote difference in velocity.

* Note the meaning we have attached to δt or δv. It is then evident that the combination of the two letters is a single symbol and not a product of two quantities, such as δ and t or δ and v.

* In the above example t denoted time. Observations are made as time progresses. So δt is always positive, whereas δv may be positive or negative. This need not always be so, as will be seen in the following illustration.

(ii) While verifying Boyles Law for a given mass of gas, the volumes and the corresponding pressures are observed at different times. Here denoting the volume by v and the pressure by p, we have $vp =$ a constant. So if the volume increases the pressure decreases and vice versa.

Let v_1 and v_2 be the volumes of the same mass when the corresponding pressures are p_1 and p_2 respectively. It then follows that if $v_2 - v_1$ is positive, then $p_2 - p_1$ is negative; if $v_2 - v_1$ is negative, then $p_2 - p_1$ is positive.

Using the difference notation we can put $v_2 - v_1 = \delta v$ and $p_2 - p_1 = \delta p$. In this case δv as well as δp can be either positive or negative. They are always opposite in sign.

(iii) In general, if x and y are two variables, one being some function of the other, and if y_1 and y_2 are values of y corresponding to the values of x_1 and x_2 of x, then we can put $x_2 - x_1 = \delta x$ and $y_2 - y_1 = \delta y$.

* $x_2 = x_1 + \delta x$ and $y_2 = y_1 + \delta y$. If $y = f(x)$ is the function, then $y_1 = f(x_1)$ and $y_2 = f(x_2)$ which also means that $y_1 + \delta y = f(x_1 + \delta x)$. Generalising this, we may say that if $y = f(x)$, then $y + \delta y = f(x + \delta x)$ i. e. $\delta y = f(x + \delta x) - y$.

$$\therefore \quad \delta y = f(x + \delta x) - f(x).$$

Let us take a concrete example. Suppose $y = f(x) \equiv x^2 + x + 1$. Take two values a and b of x. Put $a - b = \delta x$ i. e. $a = b + \delta x$. Then δy is the difference between the value of y when $x = a$ and that when $x = b$. Denote these values by $y(a)$ and $y(b)$.

$$\begin{aligned} \therefore \quad \delta y &= y(a) - y(b) \\ &= a^2 + a + 1 - (b^2 + b + 1) \\ &= f(a) - f(b) \end{aligned}$$

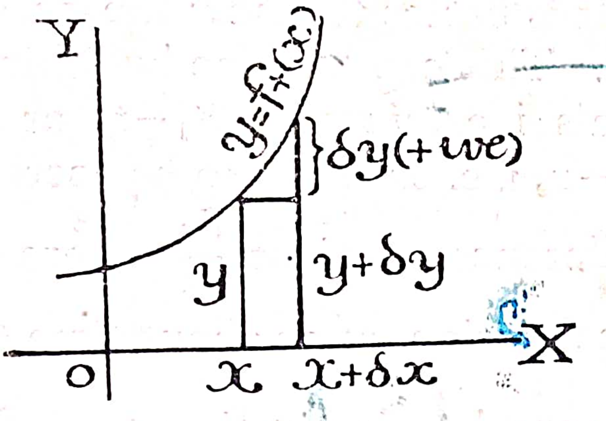
$$\therefore \quad \delta y = f(b + \delta x) - f(b).$$

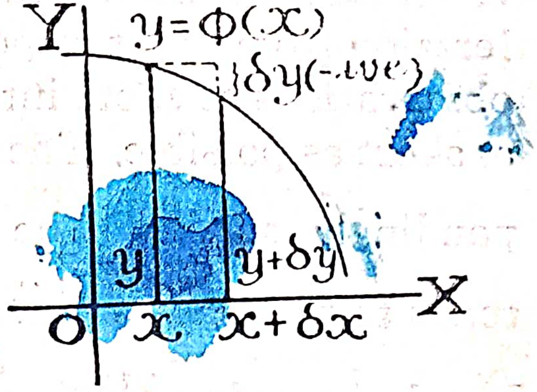
But b is any arbitrary value of x. So generally $\delta y = f(x + \delta x) - f(x)$.

* In the function $y = f(x)$. Generally x is taken as the independent variable and y as the dependent variable. So to study the nature of the function, we generally take any value of x, find out the corresponding value of y. Then we *increase x a little* ("a little" only because only thus can we study the function closely) and then find the corresponding value of y.

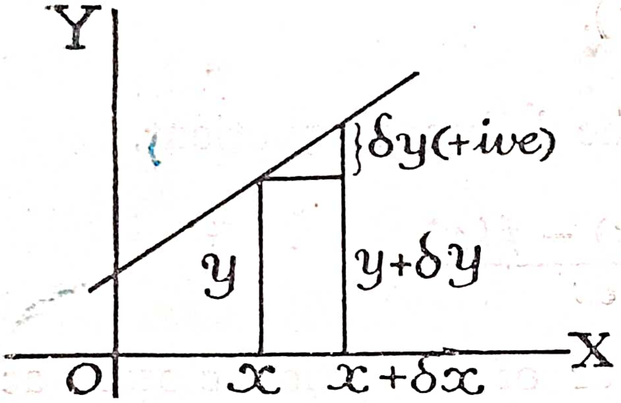
In this case therefore δx is positive whereas δy is either positive or negative.

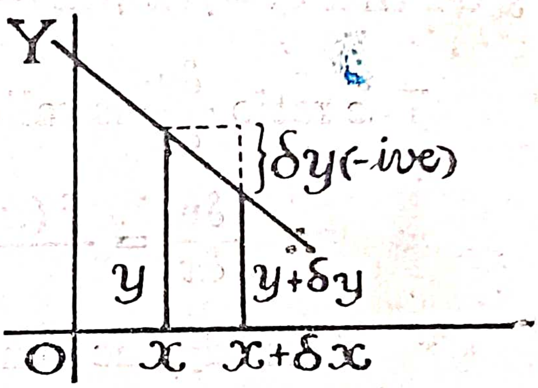
The graphs of two functions $y = (x)$ and $y = \phi(x)$ are drawn below to illustrate this point.





Observe the following linear graphs.





3·2. Difference-quotient.

(i) In the example (i) of § 3·1 the quotient $\frac{v_1 - v_0}{t_1 - t_0}$ gives us the average rate of change of velocity during the interval $t_1 - t_0$, or the average acceleration. Symbolically therefore $\frac{\delta v}{\delta t}$ stands for average acceleration in this case. $\frac{\delta v}{\delta t}$ is called the *difference-quotient* because it is the quotient of the observed differences between velocities at two particular instants and the period of time.

* If the acceleration varies from point to point, taking a large interval of time for observing the velocities and then getting the average acceleration during

the interval is not useful. It is better to know the acceleration from point to point. This can be done by taking observations at short intervals i. e. by making $t_1 - t_0$ or δt as short as possible. Then $v_1 - v_0$ i. e. δv will also be correspondingly small. The difference-quotient $\frac{\delta v}{\delta t}$ where δt tends to zero gives the acceleration at or about time t_0.

In other words, $\lim\limits_{\delta t \to 0} \frac{\delta v}{\delta t} =$ acceleration at time t_0.

(ii) Let $y = f(x)$. We saw in example (iii) of § 3·1 that $\delta y = f(x+\delta x) - f(x)$.

The ratio $\frac{\delta y}{\delta x}$ is called the difference-quotient.

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}.$$

* To study the behaviour of a function closely at any point x_1, it is necessary to make δx very small. For a small value of δx, the corresponding value of δy will also be small.

For example, let $y = x^2 + x + 1$.

When $x = 1$, $y = 3$.

Put $x = 1{\cdot}001$ (here we take $\delta x = 0{\cdot}001$)

$$\begin{aligned} \text{Then } y &= (1{\cdot}001)^2 + (1{\cdot}001) + 1 \\ &= 1{\cdot}002001 + 1{\cdot}001 + 1 \\ &= 3{\cdot}003001 \end{aligned}$$

which shows that when $\delta x = 0{\cdot}001$, $\delta y = 0{\cdot}003001$.

The difference-quotient :—

$$\begin{aligned} \frac{\delta y}{\delta x} &= \frac{0{\cdot}003001}{0{\cdot}001} \\ &= 3{\cdot}001. \end{aligned}$$

3·3. The Derivative.

We saw that if $y = f(x)$ then the value of δy depends on the value of δx. Thus δy is a function of δx. Now as $\delta x \to 0$, in most cases, $\delta y \to 0$ also. If δx is actually equal to 0, then

$$\begin{aligned}\delta y &= f(x + \delta x) - f(x)\\ &= f(x + 0) \quad - f(x)\\ &= f(x) - f(x).\end{aligned}$$

$$\therefore \quad \delta y = 0$$

so that the difference-quotient $\dfrac{\delta y}{\delta x}$ takes the form $\dfrac{0}{0}$ which is meaningless. If, however, $\delta x \to 0$, it may be possible to determine $\lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$.

The fundamental problem of differential calculus is to determine this limit, if it exists, and to interpret it in its various aspects.

Let $y = f(x)$. Give to x two values a and $a + h$, where h is small.

Here $\delta x = h$ and $\delta y = f(a + h) - f(a)$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{f(a+h) - f(a)}{h}$$

If $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists, then this limit is called the *derivative or the differential coefficient of* $f(x)$ *at* $x = a$. It is the same as $\lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$ and this limit is written as $\left(\dfrac{dy}{dx}\right)_{x=a}$ or $\dfrac{d}{dx} f(x)$ when $x = a$.

General definition of a derivative.

The limit of the difference-quotient $\frac{f(x+\delta x)-f(x)}{\delta x}$ as $\delta x \to 0$ is called the *derivative* of $f(x)$ with respect to x and is denoted by $f'(x)$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}.$$

The derivative of $f(x)$ at $x = a$ is $f'(a)$ and is obtained by putting $x = a$ in the derived function.

The various symbols $f'(x)$, $\frac{dy}{dx}$ $D_x y$ or $\frac{d}{dx} f(x)$ are used to denote the derivative of y with respect to x.

The phrase "with respect to" is written shortly as "w. r. t." We first choose δx; then find the corresponding δy Hence the phrase "with respect to".

* In all the functions that we deal with in this book, whenever $\delta x \to 0$, $\delta y \to 0$ also and the limit $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ also exists.

3·4. Examples to illustrate the method of differentiation from first principles.

(i) If $y = x^2$, find the value of $\frac{dy}{dx}$.

First: Let us find the derivative at $x = 2$.

when $x = 2$, $y = 4$.

when $x = 2 + h$, $y = (2 + h)^2$

$$= 4 + 4h + h^2.$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{4 + 4h + h^2 - 4}{h}$$

$$= 4 + h.$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{h \to 0} (4 + h)$$

$$= 4.$$

Second: Let us find the derivative at $x = a$.

when $x = a, \quad y = a^2$

when $x = a + h; \quad y = a^2 + 2ah + h^2$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{a^2 + 2ah + h^2 - a^2}{h}$$

$$= 2a + h.$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{h \to 0} (2a + h)$$

$$= 2a.$$

Third: Let us find the general expression of the derivative for any value of x.

Let x change to $x + \delta x$.

Then y changes to $y + \delta y$.

$$y = x^2$$

$$\therefore \quad y + \delta y = (x + \delta x)^2$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{(x + \delta x)^2 - y}{\delta x}$$

$$= \frac{(x + \delta x)^2 - x^2}{\delta x}$$

$$= \frac{x^2 + 2x(\delta x) + (\delta x)^2 - x^2}{\delta x}$$

$$= 2x + \delta x.$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0}, \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0}. (2x + \delta x)$$

$$= 2x.$$

Note :—When $x = 2. \ \frac{dy}{dx} = 4$ i. e. 2×2.

When $x = a \ \frac{dy}{dx} = 2a$ i. e. $2 \times a$

In general $\frac{dy}{dx} = 2x$ i. e. $2 \times x$ so it follows that at $x = 3, \ \frac{dy}{dx} = 2 \times 3$ i. e. 6 and so on.

(ii) Find the expression for the derivative of $x^2 + x + 1$ and find the value of this derivative when $x = 2$.

Solution :—Put $y = x^2 + x + 1$

Then $y + \delta y = (x + \delta x)^2 + (x + \delta x) + 1$

$$\therefore \quad \delta y = x^2 + 2x \cdot \delta x + (\delta x)^2 + x + \delta x + 1 - (x^2 + x + 1)$$

$$= 2x \cdot \delta x + (\delta x)^2 + \delta x$$

$$\therefore \quad \frac{\delta y}{\delta x} = 2x + 1 + \delta x.$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0}. \frac{dy}{dx}$$

$$= \lim_{\delta x \to 0}. (2x + 1 + \delta x)$$

$$= 2x + 1.$$

Putting $x = 2$ in $2x + 1$, we get the value of the derivative at $x = 2$ to be equal to 5.

(iii) If $y = \sqrt{x}$, find $\dfrac{dy}{dx}$.

$$y = \sqrt{x}$$

$$\therefore \quad y + \delta y = \sqrt{x + \delta x}$$

$$\therefore \quad \delta y = \sqrt{x + \delta x} - \sqrt{x}$$

$$= \frac{(\sqrt{x+\delta x} - \sqrt{x})(\sqrt{x+\delta x} + \sqrt{x})}{(\sqrt{x + \delta x} + \sqrt{x})}$$

$$= \frac{x + \delta x - x}{\sqrt{x + \delta x} + \sqrt{x}}$$

$$= \frac{\delta x}{\sqrt{x + \delta x} + \sqrt{x}}$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{1}{\sqrt{x + \delta x} + \sqrt{x}}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{1}{\sqrt{x + \delta x} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}.$$

(iv) Find, from first principles, the expression for the derivative of $ax^2 + 2hx + b$ and find the value of x for which the value of the derivative is zero.

Solution :—Put $y = ax^2 + 2hx + b$.

$$\therefore \quad y + \delta y = a(x + \delta x)^2 + 2h(x + \delta x) + b$$

$$\therefore \quad \delta y = a\,[x^2 + 2x\cdot\delta x + (\delta x)^2] + 2hx + 2h\cdot\delta x + b - (ax^2 + 2hx + b)$$

$$= 2ax\cdot\delta x + a\cdot(\delta x)^2 + 2h\cdot\delta x$$

$$\therefore \quad \frac{\delta y}{\delta x} = 2ax + a\cdot\delta x + 2h$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x\to 0} (2ax + 2h + a\cdot\delta x)$$

$$= 2ax + 2h$$

$$= 2(ax + h).$$

The value of the derivative is zero

i. e. $2(ax + h) = 0$ when $x = -\dfrac{h}{a}$.

3·5 Exercise

1. If $y = x^3$, show that $\dfrac{dy}{dx} = 3x^2$
2. If $y = x^2 - 2x$, show that $\dfrac{dy}{dx} = 2(x - 1)$
3. If $y = \dfrac{1}{x}$, show that $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$
4. If $y = \dfrac{1}{x^2}$, show that $\dfrac{dy}{dx} = -\dfrac{2}{x^3}$
5. What is the value of the derivative of $x^2 - x + 4$ when $x = 1$? Ans. 1.
6. For what value of x is the value of the derivative of $x^2 - x$ equal to 1? Ans. $x = \frac{1}{2}$

3·6 Derivatives of some standard forms

(i) *Derivative of a constant*

Let $f(x) = c$ where c is some constant.

Then $f(x + \delta x) = c$ also.

$$\therefore \quad f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{c - c}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{0}{\delta x}$$

$$= 0.$$

$\therefore$ If $y = f(x) = c$, *a* constant,

Then $\dfrac{dy}{dx} = 0.$

(ii) *Derivative of a power.*

Let $y = x^n$ where n is *a* positive integer.

Then $y + \delta y = (x + \delta x)^n$

$$\therefore \quad \delta y = (x + \delta x)^n - \delta x$$

$$= \left\{ x^n + nx^{n-1}(\delta x) + \frac{n(n-1)}{1 \cdot 2} x^{n-2} (\delta x)^2 + \ldots + (\delta x)^n \right\} - x^n$$

by the Binomial theorem for the positive integral index.

$$\therefore \quad \delta y = nx^{n-1}(\delta x) + \frac{n(n-1)}{1 \cdot 2} x^{n-2} (\delta x)^2 + \ldots\ldots\ldots$$

terms containing higher powers of δx

$$\therefore \quad \frac{\delta y}{\delta x} = nx^{n-1} + \frac{n(n-1)}{1 \cdot 2} x^{n-2} (\delta x) + \ldots\ldots\ldots\ldots\ldots$$

terms containing higher powers of δx.

Now take limits as $\delta x \to 0$. Every term of the right hand side after the first contains δx as a factor, and hence tends to the limit 0 as $\delta x \to 0$. Hence

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = nx^{n-1}$$

$\therefore$ *If* $y = x^n$ *where* n *is a positive integer, then* $\frac{dy}{dx}$ $= nx^{n-1}$

* We shall extend this result later to the case when n is any rational number. (see § 4·8)

(iii) $y = \sin x$ where x is expressed in radians.

$$\therefore\ y + \delta y = \sin(x + \delta x)$$

$$\therefore\ \delta y = \sin(x+\delta x) - y$$

$$= \sin(x+\delta x) - \sin x$$

$$\therefore\ \frac{\delta y}{\delta x} = \frac{\sin(x+\delta x) - \sin x}{\delta x}$$

$$= \frac{2\sin\frac{\delta x}{2}\cos\left(x + \frac{\delta x}{2}\right)}{\delta x}$$

$\left[\text{By the formula } \sin A - \sin B = 2\sin\frac{A-B}{2}\cos\frac{A+B}{2}\right]$

$$\therefore\ \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \left\{\frac{2\sin\frac{\delta x}{2}\cos\left(x+\frac{\delta x}{2}\right)}{\delta x}\right\}$$

$$= \lim_{\delta x \to 0} \left\{\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right\}\left[\cos\left(x+\frac{\delta x}{2}\right)\right]$$

$$= \lim_{\delta x \to 0} \left\{ \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} \right\} \times \lim_{\delta x \to 0} \cos \left(x + \frac{\delta x}{2} \right)$$

$$= 1.\cos x \left[\text{By theorem } \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \right]$$

$\therefore$ **If $y = \sin x$, then $\frac{dy}{dx} = \cos x$**

x being expressed in radians.

(iv) $y = \cos x$ where x is expressed in radians.

$$\therefore \; y + \delta y = \cos (x + \delta x)$$

$$\therefore \; \delta y = \cos (x + \delta x) - y$$

$$= \cos (x + \delta x) - \cos x$$

$$= -2 \sin \frac{\delta x}{2} \sin \left(x + \frac{\delta x}{2} \right).$$

$$\left[\text{By the formula } \cos C - \cos D = -2 \sin \frac{C-D}{2} \sin \frac{C+D}{2} \right]$$

$$\therefore \; \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-2 \sin \frac{\delta x}{2} \sin \left(x + \frac{\delta x}{2} \right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \left\{ \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} \right\} \left[- \sin \left(x + \frac{\delta x}{2} \right) \right]$$

$$= \lim_{\delta x \to 0} \left\{ \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} \right\} \times \lim_{\delta x \to 0} \left[- \sin \left(x + \frac{\delta x}{2} \right) \right].$$

$$= 1\cdot(-\sin x)\left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right].$$

$$= -\sin x$$

$\therefore$ **If $y = \cos x$ where x is expressed in radians,**

then $\frac{dy}{dx} = -\sin x$.

(v) $y = \tan x$. where x is expressed in radians.

$$\therefore \quad y + \delta y = \tan(x+\delta x)$$

$$\therefore \quad \delta y = \tan(x+\delta x) - y$$

$$= \tan(x+\delta x) - \tan x$$

$$= \frac{\sin(x+\delta x)\cos x - \cos(x+\delta x)\sin x}{\cos(x+\delta x).\cos x}$$

$$= \frac{\sin\delta x}{\cos(x+\delta x)\cdot\cos x}$$

[By the formula $\sin(A-B) = \sin A\cos B - \cos A\sin B$]

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x}$$

$$= \lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x\cdot\cos(x+\delta x)\cos x}$$

$$= \lim_{\delta x\to 0}\left[\frac{\sin\delta x}{\delta x}\right]\times\left[\frac{1}{\cos(x+\delta x)\cos x}\right]$$

$$= \lim_{\delta x\to 0}\left[\frac{\sin\delta x}{\delta x}\right]\times\lim_{\delta x\to 0}\left[\frac{1}{\cos(x+\delta x)\cos x}\right]$$

$$= 1\cdot\frac{1}{\cos^2 x}\quad\left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$$

$$= \sec^2 x$$

$\therefore$ **If $y = \tan x$ where x is expressed in radians,**

then $\dfrac{dy}{dx} = \sec^2 x$.

3·7. Exercise Differentiate from first principles (i) $\sec x$ (ii) $\operatorname{cosec} x$ (iii) $\cot x$.

Ans :—(i) $\sec x \tan x$.

(ii) $-\operatorname{cosec} x \cot x$.

(iii) $-\operatorname{cosec}^2 x$.

3·8. Illustrative Examples :—

1. Find the derivative of $\sin x^\circ$.

$$\frac{d}{dx}(\sin x^\circ) = \lim_{\delta x \to 0} \frac{\sin (x+\delta x)^\circ - \sin x^\circ}{(\delta x)^\circ}$$

$$= \lim_{\delta x \to 0} \frac{2 \sin \left(\frac{\delta x}{2}\right)^\circ \cos \left(x + \frac{\delta x}{2}\right)^\circ}{(\delta x)^\circ}$$

$$= \lim_{\delta x \to 0} \left\{ \frac{\sin \left(\frac{\delta x}{2}\right)^\circ}{\left(\frac{\delta x}{2}\right)^\circ} \right\} \times \lim_{\delta x \to 0} \cos \left(x + \frac{\delta x}{2}\right)^\circ$$

$$= \lim_{\delta x \to 0} \left\{ \frac{\sin \left(\frac{\pi \cdot \delta x}{360}\right)}{\left(\frac{\pi \cdot \delta x}{360}\right)} \right\} \times \left(\frac{\pi}{180}\right)$$

$$\times \lim_{\delta x \to 0} \cos \left(x + \frac{\delta x}{2}\right)^\circ$$

(expressing the angles in radian measure).

$$\left\{ \text{i. e. putting } \frac{1}{\left(\frac{\delta x}{2}\right)} = \frac{1}{\left(\frac{\pi \cdot \delta x}{360}\right)} \times \frac{\pi}{180} \right\}$$

$$\therefore \quad \frac{d}{dx}(\sin x^\circ) = 1 \times \frac{\pi}{180} \times \cos x$$

$$= \frac{\pi}{180} \cos x.$$

2. Differentiate $\frac{1}{x-2}$ w. r. t. x

$$\frac{d}{dx}\left[\frac{1}{x-2}\right] = \lim_{\delta x \to 0} \left[\frac{1}{x+\delta x-2} - \frac{1}{x-2}\right] \div \delta x$$

$$= \lim_{\delta x \to 0} \frac{-\delta x}{(x-2)(x+\delta x-2)} \div \delta x$$

$$= \lim_{\delta x \to 0} \frac{-1}{(x-2)(x+\delta x-2)}$$

$$= \frac{-1}{(x-2)^2}.$$

3. Differentiate $\sin 2x$ w. r. t. x

$$\frac{d}{dx}(\sin 2x) = \lim_{\delta x \to 0} \frac{\sin 2(x+\delta x) - \sin 2x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2\cos(2x+\delta x)\sin \delta x}{\delta x}.$$

$$= \lim_{\delta x \to 0} \left(\frac{\sin \delta x}{\delta x}\right) \times 2 \times \cos(2x+\delta x)$$

$$= \lim_{\delta x \to 0} \left(\frac{\sin \delta x}{\delta x}\right) \times 2 \times \lim_{\delta x \to 0} \cos(2x+\delta x)$$

$$= 1.\ 2.\ \cos 2x.$$

$$= \mathbf{2 \cos 2x}.$$

4. Differentiate $x \sin x$ w. r. t. x

$$\frac{d}{dx}(x \sin x) = \lim_{\delta x \to 0} \frac{(x+\delta x)\sin(x+\delta x) - x\sin x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{x[\sin(x+\delta x) - \sin x] + \delta x \cdot \sin(x+\delta x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2x \cos\left(x + \frac{\delta x}{2}\right) \sin\frac{\delta x}{2}}{\delta x} + \lim_{\delta x \to 0} \sin(x+\delta x)$$

$$= \lim_{\delta x \to 0} \left\{\left(\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right) \cdot x \cos\left(x + \frac{\delta x}{2}\right)\right\} + \lim_{\delta x \to 0} \sin(x+\delta x)$$

$$= 1 \cdot x \cdot \cos x + \sin x$$

$$= x \cos x + \sin x.$$

3·9. Exercises :

Differentiate w. r. t. x, from first principles

(1) $x^2 + x$ (2) $3x^2 - 4x + 1$ (3) $x^3 - x^2 + 7$

(4) $\dfrac{1}{x+4}$ (5) $(x+1)^{-1}$ ✓(6) $x \sin 2x$

(7) $\cos 3x$ (8) $\tan ax$ (9) $\dfrac{1}{x^2+1}$

(10) $x^2 \cos x$ (11) $\dfrac{\sin x}{x}$ (12) $x / \cos x$.

Answers

Ex. 3·9.

1. $2x + 1$. 2. $6x - 4$. 3. $3x^2 - 2x$.

4. $\dfrac{-1}{(x+4)^2}$. 5. $\dfrac{-1}{(x+1)^2}$. 6. $\sin 2x + 2x \cos 2x$.

7. $-3 \sin 3x$ 8. $a \sec^2 ax$. 9. $\dfrac{-2x}{(x^2+1)^2}$.

10. $2x \cos x - x^2 \sin x$. 11. $\dfrac{x \cos x - \sin x}{x^2}$.

12. $\dfrac{\cos x + x \sin x}{\cos^2 x}$.

CHAPTER IV

DIFFERENTIATION

4·1. Rules of Differentiation.

Rule I. Derivative of a sum.

Let $y = u + v$ where u and v are functions of x.

As x changes to $x + \delta x$, let u and v change to $u + \delta u$ and $v + \delta v$ respectively.

Then
$$
\begin{aligned}
y + \delta y &= (u + \delta u) + (v + \delta v) \\
&= (u + v) + (\delta u + \delta v) \\
&= y + (\delta u + \delta v)
\end{aligned}
$$

$$\therefore \quad \delta y = \delta u + \delta v.$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x}$$

$$
\begin{aligned}
\therefore \quad \frac{dy}{dx} &= \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \\
&= \lim_{\delta x \to 0} \left[\frac{\delta u}{\delta x} + \frac{\delta v}{\delta x}\right] \\
&= \lim_{\delta x \to 0} \frac{\delta u}{\delta x} + \lim_{\delta x \to 0} \frac{\delta v}{\delta x}
\end{aligned}
$$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$\therefore$ **If** $\boldsymbol{y = u + v}$**, then**

$$\boldsymbol{\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.}$$

Cor. 1. If $y = u - v$, then

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}.$$

Cor. 2. If $y = u + v + w$ ....

Where u, v, w.........are functions of x, then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \dots\dots$$

Rule II. Derivative of a Product.

Let $y = uv$ where u and v are functions of x.

As x changes to $x + \delta x$, let u and v change to $u + \delta u$ and $v + \delta v$ respectively.

$$\therefore \quad y + \delta y = (u + \delta u)\,(v + \delta v)$$

$$= uv + v\delta u + u\delta v + \delta u.\,\delta v.$$

$$= y + v\delta u + u\delta v + \delta u.\,\delta v.$$

$$\therefore \quad \delta y = v\delta u + u\delta v + \delta u.\,\delta v.$$

$$\therefore \quad \frac{\delta y}{\delta x} = v\,\frac{\delta u}{\delta x} + u\,\frac{\delta v}{\delta x} + \frac{\delta u \cdot \delta v}{\delta x}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \left(v\,\frac{\delta u}{\delta x} + u\,\frac{\delta v}{\delta x} + \delta u \cdot \frac{\delta v}{\delta x} \right)$$

$$= \lim_{\delta x \to 0} v\,\frac{\delta u}{\delta x} + \lim_{\delta x \to 0} (u + \delta u)\,\frac{\delta v}{\delta x}$$

$$= \lim_{\delta x \to 0} v \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x} + \lim_{\delta x \to 0} (u + \delta u) \times \lim_{\delta x \to 0} \frac{\delta v}{\delta x}$$

$$= v\frac{du}{dx} + u\,\frac{dv}{dx}\ \left[\because\ \delta u \to 0 \text{ as } \delta x \to 0 \right].$$

$$\therefore \quad \textbf{If } y = uv, \textbf{ then } \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}.$$

Divide the above result by y on one side and by its equivalent uv on the other. We have

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{uv}\cdot v\frac{du}{dx} + \frac{1}{uv}\cdot u\frac{dv}{dx}.$$

$\therefore$ **If $y = uv$, then**

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx}.$$

The above form is useful in extending the result thus:

Cor. 1. If $y = uvw$ where u, v, w are functions of x, then

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}.$$

The student is advised to prove this for himself.

* The result can be extended to the product of any finite number of functions.

Cor. 2. If $y = cu$ where c is a constant and u is a function of x, then

$$\frac{dy}{dx} = c\cdot\frac{du}{dx} + u\frac{dc}{dx}$$

$$= c\frac{du}{dx} + 0 \quad \text{as} \quad c \text{ is a constant.}$$

$$\therefore \quad \textbf{If } y = cu, \textbf{ then } \frac{dy}{dx} = c\frac{du}{dx}.$$

Rule III. Derivative of a quotient.

Let $y = \dfrac{u}{v}$ where u and v are functions of x.

With the usual notation and its meaning, we have

$$y + \delta y = \frac{u + \delta u}{v + \delta v}$$

$$\therefore \quad \delta y = \frac{u + \delta u}{v + \delta v} - y$$

$$= \frac{u + \delta u}{v + \delta v} - \frac{u}{v}$$

$$= \frac{uv + v\delta u - uv - u\delta v}{v\,(v + \delta v)}$$

$$= \frac{v\delta u - u\delta v}{v\,(v + \delta v)}$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{v\,\dfrac{\delta u}{\delta x} - u\,\dfrac{\delta v}{\delta x}}{v\,(v + \delta v)}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \frac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2} \qquad [\,\delta v \to 0 \text{ as } \delta x \to 0\,]$$

$$\therefore \quad \textbf{If } y = \frac{u}{v}, \textbf{ then } \frac{dy}{dx} = \frac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}$$

Cor. If $y = \dfrac{1}{v}$, then $\dfrac{dy}{dx} = -\dfrac{1}{v^2}\dfrac{dv}{dx}$

For in this case, in the relation $y = \dfrac{u}{v}$, $u = 1$

$$\therefore \quad \frac{du}{dx} = 0.$$

$$\therefore \frac{dy}{dx} = \frac{v\cdot 0 - 1\cdot\frac{dv}{dx}}{v^2}$$

$$= -\frac{1}{v^2}\frac{dv}{dx}$$

4·2 Derivatives of some standard forms (obtained by using the above rules).

1. $y = \tan x$.

(The derivative of this function was obtained in the last chapter from first principles. Here it is derived using Rule III for quotients).

$$y = \tan x = \frac{\sin x}{\cos x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos x.\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x\cdot\cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

2. $y = \text{Sec}\, x$

$$= \frac{1}{\cos x}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{\cos^2 x}\frac{d}{dx}(\cos x) \text{ by cor. of Rule III}$$

$$= \frac{-1}{\cos^2 x}(-\sin x)$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos x} \times \frac{1}{\cos x}$$

$$= \tan x . \sec x.$$

$\therefore$ **If $y = \sec x$, then $\frac{dy}{dx} = \sec x \tan x$.**

3. $y = \operatorname{cosec} x.$

$$= \frac{1}{\sin x}$$

$$\therefore \frac{dy}{dx} = \frac{-1}{\sin^2 x} \frac{d}{dx} (\sin x) \text{ by cor. of Rule III}$$

$$= \frac{-1}{\sin^2 x} \cdot \cos x.$$

$$= -\left(\frac{\cos x}{\sin x}\right) \frac{1}{\sin x}$$

$$= - \cot x \operatorname{cosec} x.$$

$\therefore$ **If $y = \operatorname{cosec} x$, then $\frac{dy}{dx} = - \operatorname{cosec} x \cot x$**

4. $y = \cot x.$

$$= \frac{\cos x}{\sin x.}$$

$$\therefore \frac{dy}{dx} = \frac{\sin x \frac{d}{dx} (\cos x) - \cos x \frac{d}{dx} (\sin x)}{\sin^2 x}$$

$$= \frac{\sin x (- \sin x) - \cos x \cdot \cos x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x}$$

$$= -\operatorname{cosec}^2 x.$$

$\therefore$ **If $y = \cot x$, then $\dfrac{dy}{dx} = -\operatorname{cosec}^2 x$.**

4·3. Illustrative Examples.

1. Differentiate w. r. t. x

$$3x^3 - 4x^2 + 7x - \frac{4}{x}$$

Put $y = 3x^3 - 4x^2 + 7x - \dfrac{4}{x}$.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(3x^3) - \frac{d}{dx}(4x^2) + \frac{d}{dx}(7x) - \frac{d}{dx}\left(\frac{4}{x}\right)$$

$$= 3\frac{d}{dx}(x^3) - 4\frac{d}{dx}(x^2) + 7\frac{d}{dx}(x) - 4\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= 3 \cdot 3x^2 - 4 \cdot 2x + 7 \cdot 1 - 4\left(\frac{-1}{x^2}\right)$$

$$= 9x^2 - 8x + 7 + \frac{4}{x^2}.$$

2. Differentiate w. r. t. x

$$(x^2 + 3)(2x - 4).$$

Put $y = (x^2 + 3)(2x - 4)$.

Let $x^2 + 3 = u$ and $2x - 4 = v$, so that $y = uv$.

$$\therefore \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[(x^2 + 3)(2x - 4)\right]$$

$$= (2x - 4)\frac{d}{dx}(x^2 + 3) + (x^2 + 3)\frac{d}{dx}(2x - 4)$$

$$= (2x-4)\left[\frac{d}{dx}(x^2)+\frac{d}{dx}(3)\right]+(x^2+3)\left[\frac{d}{dx}(2x)-\frac{d}{dx}(4)\right]$$

$$= (2x-4)(2x+0)+(x^2+3)(2+0)$$

$$= (2x-4)\,2x+(x^2+3)\,2$$

$$= 4x^2-8x+2x^2+6.$$

$$= 6x^2-8x+6.$$

* The derivative can also be found out by first multiplying out $(x^2+3)(2x-4)$ and then differentiating term by term.

3. If $y=(1+x)(1+x^2)\sin x$, find $\frac{dy}{dx}$.

Let $1+x=u$; $1+x^2=v$; $\sin x=w$ where u, v, w are functions of x.

$$\therefore \quad \frac{1}{y}\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}$$

$$\therefore \quad \frac{1}{(1+x)(1+x^2)\sin x}\cdot\frac{dy}{dx}=\frac{1}{1+x}\frac{d}{dx}(1+x)$$

$$+\frac{1}{1+x^2}\frac{d}{dx}(1+x^2)+\frac{1}{\sin x}\frac{d}{dx}(\sin x)$$

$$=\frac{1}{1+x}\cdot(0+1)+\frac{1}{1+x^2}(0+2x)+\frac{1}{\sin x}\cdot\cos x$$

$$=\frac{1}{1+x}+\frac{2x}{1+x^2}+\cot x.$$

$$\therefore \quad \frac{dy}{dx}=(1+x)(1+x^2)\sin x\left[\frac{1}{1+x}+\frac{2x}{1+x^2}+\cot x\right]$$

4. If $y = \dfrac{x^2}{1-x^2}$ find $\dfrac{dy}{dx}$.

Let $x^2 = u$; $(1 - x^2) = v$ where u and v are functions of x.

$$\therefore \quad y = \frac{u}{v}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(1-x^2)\dfrac{d}{dx}(x^2) - x^2\dfrac{d}{dx}(1-x^2)}{(1-x^2)^2}$$

$$= \frac{(1-x^2)\,2x - x^2(-2x)}{(1-x^2)^2}$$

$$= \frac{2x - 2x^3 + 2x^3}{(1-x^2)^2}$$

$$= \frac{2x}{(1-x^2)^2}.$$

5. If $y = \dfrac{\cos x + \sin x}{\cos x - \sin x}$, find $\dfrac{dy}{dx}$.

Put $u = \cos x + \sin x$ and

$v = \cos x - \sin x$.

$$\therefore \quad y = \frac{u}{v}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(\cos x - \sin x)\dfrac{d}{dx}(\cos x + \sin x) - (\cos x + \sin x)\dfrac{d}{dx}(\cos x - \sin x)}{(\cos x - \sin x)^2}$$

$$= \frac{(\cos x - \sin x)(-\sin x + \cos x) - (\cos x + \sin x)(-\sin x - \cos x)}{(\cos x - \sin x)^2}$$

$$= \frac{(\cos x - \sin x)^2 + (\cos x + \sin x)^2}{(\cos x - \sin x^2)^2}$$

$$= \frac{2\,(\cos^2 x + \sin^2 x)}{(\cos x - \sin x)^2}$$

$$= \frac{2}{(\cos x - \sin x)^2}.$$

6. If $y = \sec x + \tan x$, show that $\frac{dy}{dx} = y \sec x$.

$$y = \sec x + \tan x.$$

$$\therefore \quad \frac{dy}{dx} = \sec x.\tan x + \sec^2 x.$$

$$= \sec x\,(\tan x + \sec x).$$

$$= \sec x \times y.$$

$$\therefore \quad \frac{dy}{dx} = y \sec x.$$

4·4. Exercises.

Differentiate *w. r. t.* x the following.

(1) $3 + 2x + 4x^2$

(2) $5 - \frac{6}{x} + \frac{1}{x^2}$

(3) $2x^3 - 9x^2 + 12x - 3$

(4) $x^4 - 4x^3 + 4x^2 - 3$

(5) $(x+1)(x+2)$

(6) $(x^2+1)(2x-3)$

(7) $(x^2+x-1)(x^2-x+1)$

(8) $x \cos x - x^2 \sin x$

(9) $(1+x)(1+x^2)(1+x^3)$

(10) $x \sin x \cos x$

(11) $\frac{1+x}{1-x}$

(12) $\frac{\sqrt{x}}{x-1}$

(13) $\frac{\sin x}{3 + \cos x}$

(14) $\dfrac{a+b\cos x}{b+a\cos x}$ (15) $\dfrac{1+\tan x}{1-\tan x}$

(16) $\dfrac{2x+5}{x^2-3x+2}$ (17) $\dfrac{x+1}{(x-1)(x-2)}$

(18) If $y=\cos x+\sin x$, find the value of x for which

$$\frac{dy}{dx}=y.$$

(19) If $y=\dfrac{1}{(2x+1)(1-x)}$ for what value of x is

$$\frac{dy}{dx}=0?$$

(20) If $t=\tan\dfrac{x}{2}$, differentiate $\sin x$, $\cos x$, $\tan x$ and $\sec^2 x$ w. r. to t. Express the results in terms of t.

4·5. Differentiation of a function of a function.

Suppose y is a function of u where u itself is a function of x. y is such a function of u that you can differentiate y. *w. r. t. u.* Again u is such a function of x that you can differentiate u. *w. r. t. x.* We however, want to know how to differentiate y. *w. r. t. x*, where y is a *function of a function.*

Let $y=f(u)$ and $u=\phi(x)$ so that y is a function of x expressed as

$$y=f\{\phi(x)\}$$

As x changes to $x+\delta x$, let u change to $u+\delta u$ and let this change in u make y change to $y+\delta y$. Then we have the identical algebraic relation

$$\frac{\delta y}{\delta x}=\frac{\delta y}{\delta u}\cdot\frac{\delta u}{\delta x} \text{ provided } \delta u\neq 0.$$

Now as u is a differentiable function of x,

$$\therefore \quad \delta u \to 0 \text{ as } \delta x \to 0$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x}$$

$$= \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta u}\right) \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x}\right)$$

$$= \lim_{\delta u \to 0} \left(\frac{\delta y}{\delta u}\right) \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x}\right)$$

$$= \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\therefore \boldsymbol{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

*1. In particular if $y = x, \dfrac{dy}{dx} = 1,$

$$\therefore \quad \frac{dy}{du} \cdot \frac{du}{dx} = 1 \text{ i. e. } \frac{dx}{du} \cdot \frac{du}{dx} = 1$$

$$\therefore \quad \frac{du}{dx} = \frac{1}{\frac{dx}{du}} \text{ if } \frac{dx}{du} \neq 0$$

Replacing u by y, we have if $\dfrac{dx}{dy} \neq 0$, then

$$\boldsymbol{\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}}$$ **which is a very important relation.**

*2. If $\dfrac{du}{dx} \neq 0$, we may write the result

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \text{ as}$$

$$\frac{dy}{du} = \frac{\frac{dy}{dx}}{\frac{du}{dx}}.$$

* 3. We may generalise the result of this article. Thus if y is a function of u, where u is a function of v and v is a function of x,

$$\text{Then } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

4·6. Illustrative examples :

1. If $y = (ax + b)^4$, find $\dfrac{dy}{dx}$.

Put $ax + b = u$.

Then $y = u^4$ i. e. y is a function of u where u is a function of x.

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{du}(u^4) \cdot \frac{d}{dx}(ax + b)$$

$$= 4u^3 \times a$$

$$= 4a(ax + b)^3.$$

2. If $y = \sin(x^2 + 1)$ find $\dfrac{dy}{dx}$.

Put $x^2 + 1 = u$.

Then $y = \sin u$ i. e. y is a function of u, where u is a function of x.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{du}(\sin u) \cdot \frac{d}{dx}(x^2 + 1)$$

$$= \cos u \cdot 2x$$

$$= 2x \cos (x^2 + 1).$$

3. Differentiate $\cos^2 x$ w. r. t. x

Put $\quad y = \cos^2 x$

and $\quad u = \cos x$

$\therefore \quad y = u^2$, a function of a function.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{du}(u^2) \cdot \frac{d}{dx}(\cos x)$$

$$= 2u \cdot (-\sin x)$$

$$= -2 \sin x \cos x$$

$$= -\sin 2x.$$

4. If $\quad y = \dfrac{\sin x^2}{x^2}$ find $\dfrac{dy}{dx}$.

Put $\quad u = \sin x^2$

and $\quad v = x^2$

$$\therefore \quad y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v \dfrac{du}{dx} - u \dfrac{dv}{dx}}{v^2}$$

If $\quad u = \sin x^2, \quad \dfrac{du}{dx} = 2x \cos x^2.$

Also $\quad v = x^2 \quad \therefore \quad \dfrac{dv}{dx} = 2x.$

$$\therefore \quad \frac{dy}{dx} = \frac{x^2 \cdot 2x \cos x^2 - \sin x^2 \cdot 2x}{x^4}$$

$$= \frac{2x^3 \cos x^2 - 2x \sin x^2}{x^4}$$

$$= \frac{2x\,(x^2 \cos x^2 - \sin x^2)}{x^3}.$$

5. If $y = \sin^2 x^3$ find $\frac{dy}{dx}$.

Put $u = \sin x^3$

and $v = x^3$

$\therefore \quad y = u^2$ where $u = \sin v$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= \frac{d}{du}(u^2) \cdot \frac{d}{dv}(\sin v) \cdot \frac{d}{dx}(x^3)$$

$$= 2u \cdot \cos v \cdot 3x^2$$

$$= 2 \cdot \sin x^3 \cdot \cos x^3 \cdot 3x^2$$

$$= 6x^2 \sin 2x^3.$$

4·7 Exercises:

Differentiate the following.

(1) $(3x+4)^3$ (2) $(2x-3)^5$

(3) $(x^2+x+1)^2$ (4) $\frac{1}{x^2+1}$

(5) $\frac{x^2-1}{x^2+1}$ (6) $\cos(x^2-1)$

(7) $\tan(x^2+x-3)$ (8) $\tan\left(\frac{x-1}{x+1}\right)$

(9) $x^3 \cos^2 x$ (10) $\sin x + \sin^2 x$

(11) $\dfrac{\sin x}{\cos^3 x}$ (12) $\sin x \sin x^2$

(13) $\dfrac{x^3 \cos^2 x}{\sin x}$ (14) $\dfrac{(1+x)(1+x^2)}{1+x^3}$

(15) $\tan \dfrac{x}{2}$ (16) $\sin^2 \dfrac{x}{2}$

(17) $\cos 2x \sin 4x$ (18) $x^3 \sec^2 3x$

(19) $(x^3-1) \tan^2 \dfrac{x}{2}$ (20) $\tan^2 \dfrac{x}{2} \sec^2 \dfrac{x}{2}$

(21) $\left(x+\dfrac{1}{x}\right)^2$ (22) $\left(\dfrac{2x+1}{x+1}\right)^3$

(23) $\sqrt{1+\sin x}.\ \sec \dfrac{x}{2}$ (24) $\dfrac{-x^2}{(x^2+1)^2}$

(25) $x^3 \sin^2 x \cos x.$ (26) $\dfrac{1-\cos^2 x}{1+\cos^2 x}$

(27) $\sin^2\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)$ (28) $\dfrac{\sin \dfrac{x}{2}+\cos \dfrac{x}{2}}{\sin \dfrac{x}{2}-\cos \dfrac{x}{2}}$

(29) $\tan\left(\dfrac{ax+b}{bx+a}\right)$ (30) $\sec(x^\circ+45^\circ)$

4·8 To prove that the derivative of x^n is nx^{n-1} for all rational values of n

Let $y = x^n$ where n is a rational number and x^n is real.

Case (i) When n is a positive integer we have seen that $\dfrac{dy}{dx} = nx^{n-1}$. [see § 3·6 (ii)]

Case (ii) When n is a negative integer.

Let $n = -m$ where m is a positive integer.

Then $y = x^n = x^{-m} = \dfrac{1}{x^m}$

Put $z = x^m$ so that $\dfrac{1}{x^m} = \dfrac{1}{z}$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$$

$$= \frac{d}{dz}\left(\frac{1}{z}\right)\frac{d}{dx}(x^m)$$

$$= -\frac{1}{z^2}\cdot mx^{m-1}$$

$$= -\frac{1}{(x^m)^2}\cdot mx^{m-1}$$

$$= \frac{-1}{x^{2m}}\cdot mx^{m-1}$$

$$= -m\,x^{m-1-2m}$$

$$= -m\,x^{-m-1}$$

$$= (-m)\,x^{(-m)-1}$$

$$= n\,x^{n-1} \text{ putting } n \text{ for } -m.$$

Case (iii) Let $n = p/q$ where p is any integer, positive or negative and q is a positive integer.

Here $y = x^n = x^{p/q} = (x^{1/q})^p$

Put $u = x^{1/q}$

Then $y = u^p$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{dy}{du}\bigg/\frac{dx}{du}. \qquad \text{Cor 2 of § 4·5}$$

Now $y = u^p \quad \therefore \quad \dfrac{dy}{du} = pu^{p-1}$ by cases (i) and (ii)

and $u = x^{1/q} \quad \therefore \quad x = u^q$

$$\therefore \quad \frac{dx}{du} = qu^{q-1}$$

$$\therefore \quad \frac{dy}{dx} = \frac{pu^{p-1}}{qu^{q-1}}$$

$$= \frac{p}{q}\, u^{p-q}$$

$$= \frac{p}{q}\, (x^{1/q})^{p-q}$$

$$= \frac{p}{q}\, x^{p/q-1}$$

$$= n\, x^{n-1} \text{ putting } n = \frac{p}{q}.$$

$\therefore$ **If $y = x^n$, then $\dfrac{dy}{dx} = n\,x^{n-1}$ where n is any rational number.**

4·9. Illustrative examples.

1. $y = \sqrt{x} = x^{1/2}$

$$\therefore \quad \frac{dy}{dx} = \tfrac{1}{2}\, x^{1/2-1} = \tfrac{1}{2}\, x^{-1/2} = \frac{1}{2\sqrt{x}}$$

2. $y = (ax + b)^{3/2}$.

Put $ax + b = u$

$$\therefore \quad y = u^{3/2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{du}{dx}$$

$$= \frac{3}{2}\, v^{3/2-1} \cdot \frac{d}{dx}(ax + b)$$

$$= \frac{3}{2} v^{1/2};\ a.$$

$$= \frac{3a}{2} (ax + b)^{1/2}$$

3. $y = \sin^m (n\theta)$ where m is any rational number.

Put $u = \sin n\theta$ and $v = n\theta$.

$\therefore\quad y = u^m$ where $u = \sin v$

and $v = n\theta$.

$$\therefore\quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{d\theta}$$

$$= \frac{d}{du} (u^m) \cdot \frac{d}{dv} (\sin v) \frac{d}{d\theta} (n\theta)$$

$$= m\, u^{m-1}.\ \cos v.\ n.$$

$$= mn \sin^{m-1} (n\theta) \cos (n\theta).$$

4. $y = \dfrac{px + q}{\sqrt{ax^2 + 2hx + b}}$

Put $u = px + q;\ v = \sqrt{ax^2 + 2hx + b}.$

$$\therefore\quad y = \frac{u}{v}.$$

$$\therefore\quad \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{\sqrt{ax^2 + 2hx + b}.\ p - (px+q)\dfrac{dv}{dx}}{ax^2 + 2hx + b}$$

Now $v = \sqrt{ax^2 + 2hx + b}$

$= w^{1/2}$ where $w = ax^2 + 2hx + b.$

$$\therefore \quad \frac{dv}{dx}=\frac{dv}{dw}\cdot\frac{dw}{dx}$$

$$=\frac{d}{dw}\,(w^{1/2})\cdot\frac{d}{dx}\,(ax^2+2hx+b)$$

$$=\frac{1}{2}\,w^{-1/2}\cdot(2ax+2h)$$

$$=\frac{ax+h}{w^{\frac{1}{2}}}$$

$$=\frac{ax+h}{\sqrt{ax^2+3hx+b}}$$

$$\therefore \quad \frac{dy}{dx}=\frac{p\sqrt{ax^2+2hx+b}-\dfrac{(px+q)\,(ax+h)}{\sqrt{ax^2+2hx+b}}}{ax^2+2hx+b}$$

$$=\frac{p\,(ax^2+2hx+b)-(px+q)(ax+h)}{(ax^2+2hx+b)^{3/2}}$$

$$=\frac{x\,(2hp-aq-bp)+b\,(p-q)}{(ax^2+2hx+b)^{3/2}}$$

After some practice, the student can dispense with writing down the actual substitution and make the substitution mentally.

5. The following examples will illustrate this:

$$y=\frac{x}{\sqrt{a^2+x^2}}$$

$$\therefore \quad \frac{dy}{dx}=\frac{\sqrt{a^2+x^2}\cdot 1-x\cdot\dfrac{\frac{1}{2}}{\sqrt{a^2+x^2}}\cdot 2\,x}{(a^2+x^2)}$$

$$=\frac{(a^2+x^2)-x^2}{(a^2+x^2)^{3/2}}$$

$$=\frac{a^2}{(a^2+x^2)^{3/2}}$$

6. $y = \sqrt{\cos 2\sqrt{x}}$

$$\frac{dy}{dx} = \tfrac{1}{2}\,\frac{1}{\sqrt{\cos 2\sqrt{x}}}\,(-\sin 2\sqrt{x})\cdot 2\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x}}$$

$$= \frac{-\sin 2\sqrt{x}}{2\sqrt{x}\cos 2\sqrt{x}}$$

4·10 Exercises :—

Differentiate the following

(1) $x^{3/2}$

(2) $\dfrac{1}{\sqrt{x}}$

(3) $\dfrac{1}{x^{2/3}}$

(4) $(x-1)^{-3}$

(5) $(3x+4)^{3/4}$

(6) $\dfrac{x}{\sqrt{x+1}}$

(7) $\dfrac{\sqrt{x^2+1}}{x}$

(8) $(1+x)\,(1+x^2)^{-1}$

(9) $\dfrac{1}{\sqrt{3x^2+4x+1}}$

(10) $x\sqrt{x^2-4}$

(11) $\sqrt{\sin 2x}$

(12) $\sqrt{\sec x \tan x}$

(13) $[x-\sqrt{1-x^2}\,]^2$

(14) $\sqrt{x}\,(1-\sqrt{x})$

(15) $\dfrac{\sqrt{a^2-x^2}}{x}$

(16) $\dfrac{x}{\sqrt{x^2-4}}$

(17) $\sqrt{x^2-a^2}+\dfrac{a^2}{\sqrt{x^2-a^2}}$

(18) $\sqrt{1+k^2\tan^2 x}$

(19) $\sqrt{a+b\tan^2 x}$

(20) $\sin\sqrt{x}$

(21) $\sqrt{\sin\sqrt{x}}$

(22) $\sqrt{(x-a)\,(x-b)}$

(23) $\dfrac{\sqrt{\sin x}}{\cos x}$ (24) $\dfrac{\sqrt{\cos x}}{\cos \sqrt{x}}$

(25) $\sqrt{\dfrac{x}{x-1}}$ (26) $\sqrt{\dfrac{ax^2+2bx+c}{Ax^2+2Bx+C}}$

(27) $\sqrt{\dfrac{x-1}{x+1}}$ (28) $\sin^p x . \cos^q x$

(29) $\dfrac{1}{\sqrt{x}+\sqrt{x+1}}$ (30) $\sqrt{a \sin^2 x + b \cos^2 x}$

4·11. Derivatives of inverse trigonometric functions.

The use of the important Cor. 1 of § 4·5 is illustrated in the following examples :

1. If $y = \sin^{-1} x$ find $\dfrac{dy}{dx}$.

$$y = \sin^{-1} x \quad \therefore \quad x = \sin y.$$

$$\therefore \quad \frac{dx}{dy} = \cos y.$$

$$\text{But} \quad \frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}}$$

$$= \frac{1}{\sqrt{1-x^2}}.$$

$\therefore$ **If $y = \sin^{-1} x$, then $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$.**

2. If $y = \cos^{-1} x$, find $\dfrac{dy}{dx}$.

$$y = \cos^{-1} x \quad \therefore \quad x = \cos y$$

$$\therefore \quad \frac{dx}{dy} = -\sin y$$

But $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{-\sin y}$$

$$= \frac{-1}{\sqrt{1-\cos^2 y}}$$

$$= \frac{-1}{\sqrt{1-x^2}}$$

$\therefore$ **If $y = \cos^{-1} x$, then $\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$.**

3. If $y = \tan^{-1} x$, find $\frac{dy}{dx}$.

$$y = \tan^{-1} x \quad \therefore \quad x = \tan y$$

$$\therefore \quad \frac{dx}{dy} = \sec^2 y$$

$$= 1 + \tan^2 y$$

$$= 1 + x^2.$$

But $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{1+x^2}$$

$\therefore$ **If $y = \tan^{-1} x$, then $\frac{dy}{dx} = \frac{1}{1+x^2}$**

We can similarly obtain the following :

If $y = \cot^{-1} x$, then $\frac{dy}{dx} = \frac{-1}{1+x^2}$

If $y = \sec^{-1} x$, then $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$

If $y = \operatorname{cosec}^{-1} x$, then $\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$

4·12. Illustrative examples :

Ex. 1. If $y = \sin^{-1}\frac{x}{a}$, find $\frac{dy}{dx}$

Put $u = \frac{x}{a}$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{a}$$

$$= \frac{1}{\sqrt{1-\frac{x^2}{a^2}}} \cdot \frac{1}{a}$$

$$= \frac{1}{\sqrt{a^2 - x^2}}.$$

Ex. 2. If $y = \tan^{-1}\frac{x}{u}$, find $\frac{dy}{dx}$.

Put $u = \frac{x}{a}$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{1+u^2} \cdot \frac{1}{a}$$

$$= \frac{1}{1+\frac{x^2}{a^2}} \cdot \frac{1}{a}$$

$$= \frac{a}{a^2 + x^2}.$$

Ex. 3. $y = x \cos^{-1} x$.

Put $u = x$ and $v = \cos^{-1} x$

$\therefore \quad y = uv$.

$$\therefore \quad \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$= \cos^{-1} x \;\; 1 + x \frac{-1}{\sqrt{1-x^2}}$$

$$= \cos^{-1} x - \frac{x}{\sqrt{1-x^2}}.$$

Ex. 4. If $y = \tan^{-1}\left(\frac{1-x}{1+x}\right)$ find $\frac{dy}{dx}$.

Method I. Put $\frac{1-x}{1+x} = u$.

$$\therefore \quad y = \tan^{-1} u.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{1+u^2} \cdot \frac{(1+x)(-1)-(1-x)\cdot 1}{(1+x)^2}$$

$$= \frac{1}{1+\left(\frac{1-x}{1+x}\right)^2} \cdot \frac{-2}{(1+x)^2}$$

$$= \frac{-2}{(1+x)^2+(1-x)^2}$$

$$= \frac{-1}{1+x^2}$$

Method II. $y = \tan^{-1}\frac{1-x}{1+x} = \tan^{-1}\frac{1-x}{1+1\cdot x}$

Put $1 = \tan\frac{\pi}{4}$ and $x = \tan\theta$.

$$\therefore \quad y = \tan^{-1}\left\{\frac{\tan\frac{\pi}{4} - \tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right\}$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\theta\right)\right]$$

$$\therefore \quad y = \frac{\pi}{4} - \theta.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$= -1\cdot\frac{d}{dx}(\tan^{-1}x) \qquad \because \theta = \tan^{-1}x$$

$$= \frac{-1}{1+x^2}.$$

Ex. 5. If $y = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$ find $\dfrac{dy}{dx}$.

Put $x = \sin\theta$

$$\therefore \quad y = \tan^{-1}\left[\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right]$$

$$= \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$= \tan^{-1}(\tan\theta)$$

$$\therefore \quad y = \theta.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$= 1\cdot\frac{d}{dx}(\sin^{-1}x)$$

$$= \frac{1}{\sqrt{1-x^2}}.$$

Ex. 6. If $y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ find $\frac{dy}{dx}$

Put $x = \frac{\pi}{2} - \theta.$

$$\therefore \quad y = \tan^{-1}\left\{\frac{\cos\left(\frac{\pi}{2}-\theta\right)}{1+\sin\left(\frac{\pi}{2}-\theta\right)}\right\}$$

$$= \tan^{-1}\left[\frac{\sin\theta}{1+\cos\theta}\right].$$

$$= \tan^{-1}\left\{\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$\therefore \quad y = \frac{\theta}{2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$= \frac{1}{2}\cdot\frac{d}{dx}\left(\frac{\pi}{2}-x\right)$$

$$= \frac{1}{2}(-1) = -\frac{1}{2}.$$

4·13 Exercises:—

Differentiate the following.

(1) $\cos^{-1}\frac{x}{a}$

(2) $\cot^{-1}\frac{x}{a}$

(3) $\sin^{-1}x^2$

(4) $\tan^{-1}\sqrt{x}$

(5) $\sec^{-1} 2x$ (6) $\sin^{-1}\left(\frac{1}{x}\right)$

(7) $\sec^{-1}\sqrt{1+x^2}$ (8) $\cos^{-1}\sqrt{1-x^2}$

(9) $\tan^{-1}\left(\frac{1+x}{1-x}\right)$ (10) $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$

(11) $\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ (12) $x\tan^{-1} x$

(13) $\sin^{-1}(\cos x)$ (14) $\cos^{-1}(\sin x)$

(15) $\tan^{-1}(\cot x)$ (16) $(1+x^2)\tan^{-1} x - x$

(17) $\sqrt{1-x^2}\sin^{-1} x - x$

(18) $\sin^{-1} x + \sin^{-1}\sqrt{1-x^2}$

(19) $\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$ (20) $\cos^{-1}(2\cos\theta)$

(21) If $y = \sqrt{1+x^2}\cos(\tan^{-1} x)$, prove that $\frac{dy}{dx} = 0$.

4·14 Miscellaneous Examples.

Ex. 1. Consider the function $x^2+y^2=a^2$. Here y is a function of x and vice versa. For, given a particular value of x, we can find the corresponding value of y and vice versa. To find $\frac{dy}{dx}$, we proceed as follows:—

$$x^2+y^2=a^2$$

$$\therefore \quad y^2=a^2-x^2$$

$$\therefore \quad y=\sqrt{a^2-x^2}$$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{2}\,\frac{-2x}{\sqrt{a^2-x^2}}$$

$$=\frac{-x}{\sqrt{a^2-x^2}}$$

$$=\frac{-x}{y}$$

Compare the above method with the following method.

Alternate Method.

Take $x^2+y^2=a^2$ and differentiate every term on both sides w. r. t. x. The derivative of one side must be equal to that of the other of the two sides of an equation.

$$\therefore \quad \frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=\frac{d}{dx}(a^2)$$

Remember that y is a function of x

$$\therefore \quad \frac{d}{dx}(y^2)=\frac{d}{dy}(y^2)\,\frac{d}{dx}(y)$$

$$=2y\cdot\frac{dy}{dx}$$

Because y^2 is a function of y where y itself is a function of x, therefore y^2 is a function of a function.

$\therefore$ The above equation becomes

$$2x+2y\frac{dy}{dx}=0 \quad (\because \ a^2 \text{ is a constant})$$

$$\therefore \quad \frac{dy}{dx}=\frac{-x}{y}$$

The advantage of the above method will be seen clearly in the following example.

Ex. 2. If $ax^2+2hxy+by^2=0$ find $\dfrac{dy}{dx}$.

From the equation, it follows that y is a function of x; but it is difficult to express the value of y in terms of x alone. So to find $\dfrac{dy}{dx}$ we proceed to differentiate both sides of the equation as below, w. r. t. x

$$\frac{d}{dx}(ax^2)+\frac{d}{dx}(2hxy)+\frac{d}{dx}(by^2)=\frac{d}{dx}(0)$$

$$a\cdot 2x+2h\frac{d}{dx}(xy)+b\frac{d}{dx}(y)^2=0$$

$$2ax+2h\left[x\cdot\frac{dy}{dx}+y\cdot\frac{d}{dx}(x)\right]+b\cdot\frac{d}{dy}(y^2)\frac{dy}{dx}=0$$

$$2ax+2h\left[x\frac{dy}{dx}+y\cdot 1\right]+b\cdot 2y\cdot\frac{dy}{dx}=0$$

$$2ax+2hx\frac{dy}{dx}+2hy+2by\frac{dy}{dx}=0$$

$$\therefore\quad \frac{dy}{dx}=-\frac{ax+hy}{hx+by}.$$

Ex. 3. We work out again illustrative example 4 of §·4·9 to demonstrate how the new idea that we have introduced in this article leads to ease in calculations.

If $y=\dfrac{px+q}{\sqrt{ax^2+2hx+b}}$ find $\dfrac{dy}{dx}$.

Squaring both sides, we have

$$y^2=\frac{(px+q)^2}{ax^2+2hx+b}$$

$$\therefore\quad y^2(ax^2+2hx+b)=(px+q)^2$$

$$\therefore\quad (ax^2+2hx+b)\,2y\frac{dy}{dx}+y^2\cdot(2ax+2h)$$
$$=2(px+q)\cdot p.$$

$$\therefore\quad (ax^2+2hx+b)\,y\frac{dy}{dx}=p(px+q)-y^2(ax+h)$$

$$\therefore\quad \frac{dy}{dx}=\frac{p(px+q)-y^2(ax+h)}{y(ax^2+2hx+b)}$$

$$=\frac{p\,(px+q)-\dfrac{(px+q)^2}{ax^2+2hx+b}\,(ax+h)}{\dfrac{(px+q)}{\sqrt{ax^2+2hx+b}}\cdot(ax^2+2hx+b)}$$

$$=\frac{p-\dfrac{(px+q)(ax+h)}{ax^2+2hx+b}}{\sqrt{ax^2+2hx+b}}$$

$$=\frac{p\,(ax^2+2hx+b)-(px+q)\,(ax+h)}{(ax^2+2hx+b)^{3/2}}$$

$$=\frac{x\,(2hp-aq-bp)+b\,(p-q)}{(ax^2+2hx+b)^{3/2}}.$$

4·15. Exercises:—

Find $\dfrac{dy}{dx}$ in the following cases:—

(1) $x^2+y^2=5$

(2) $2x^2+3y^2=4$

(3) $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$

(4) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

(5) $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

(6) $xy=c^2$

(7) $x^2+2xy+y^2=0$

(8) $x^2-xy+7y^2=0.$

(9) $x^2+xy-3y^2+4x+3y+5=0.$

(10) $ax^2+2hxy+by^2+2gx+2fy+c=0.$

Answers

Ex. 4·4

1. $2+8x$
2. $\dfrac{6}{x^2}-\dfrac{2}{x^3}$
3. $6x^2-18x+12$
4. $4x^3-12x^2+8x$
5. $2x+3$
6. $6x^2-6x+2$

7. $4x^3-2x+2$

8. $\cos x\,(1-x^2)-x\sin x\,(1+2x)$

9. $6x^5+5x^4+4x^3+6x^2+2x+1$

10. $\frac{1}{2}\sin 2x+x\cos 2x$

11. $\dfrac{2}{(1-x)^2}$

12. $\dfrac{-(1+x)}{2\sqrt{x}\,(1-x)^2}$

13. $\dfrac{3\cos x+1}{(3+\cos x)^2}$

14. $\dfrac{(a^2-b^2)\sin x}{(b+a\cos x)^2}$

15. $\dfrac{2\sec^2 x}{(1-\tan x)^2}$

16. $\dfrac{-2x^2-10x+19}{(x^2-3x+2)^2}$

17. $\dfrac{x^2-2x+5}{(x-1)^2\,(x-2)^2}$

18. $n\pi$ where n is any integer

19. $\frac{1}{4}$

20. $\dfrac{2(1-t^2)}{(1+t^2)^2},\ \dfrac{-4t}{(1+t^2)^2},\ \dfrac{2(1+t^2)}{(1-t^2)^2},\ \dfrac{1-2t^4}{(1-t^2)^4}$

Ex. 4·7

1. $9(3x+4)^2$

2. $10(2x-3)^4$

3. $2(x^2+x+1)(2x+1)$

4. $\dfrac{-2x}{(x^2+1)^2}$

5. $\dfrac{4x}{(x^2+1)^2}$

6. $-2x\sin(x^2-1)$

7. $(2x+1)\sec^2(x^2+x-3)$

8. $\dfrac{2}{(x+1)^2}\sec^2\left(\dfrac{x-1}{x+1}\right)$

9. $3x^2\cos^2 x-x^3\sin 2x$

10. $\cos x\,(1+2\sin x)$

11. $\sec^2 x\,(3\tan^2 x+1)$

12. $\cos x\sin x^2+2x\sin x\cdot\cos x^2$

13. $\dfrac{3x^2\cos^2 x}{\sin x}-\dfrac{x^3\cos x\,(1+\sin^2 x)}{\sin^2 x}$

14. $\dfrac{1-x^2}{(1-x+x^2)^2}$

15. $\dfrac{1}{2}\sec^2\dfrac{x}{2}$

16. $\frac{\sin x}{2}$

17. $3\cos 6x+\cos 2x$

18. $3x^2\sec^2 3x\,(1+2x\tan 3x)$

19. $3x^2\tan^2\frac{x}{2}+(x^3-1)\tan\frac{x}{2}\sec^2\frac{x}{2}$

20. $\tan\frac{x}{2}\left(1+\tan^2\frac{x}{2}\right)\left(1+2\tan^2\frac{x}{2}\right)$

21. $2x-\frac{2}{x^3}$

22. $\frac{3(2x+1)}{(x+1)^3}$

23. $\frac{1}{2}\sec^2\frac{x}{2}\left(\text{use }\sqrt{1+\sin x}=\sin\frac{x}{2}+\cos\frac{x}{2}\right)$

24. $\frac{2x(1-x^2)}{(1+x^2)^3}$

25. $x^3\sin x\,(2\cos x-1)+3x^2\cos x\,(1-\cos x)$

26. $\frac{2\sin 2x}{(1+\cos^2 x)^2}$

27. $\frac{1}{2}\cos x$

28. $\frac{1}{\sin x-1}$

29. $\frac{a^2-b^2}{(bx+a)^2}\sec^2\left(\frac{ax+b}{bx+a}\right)$

30. $\frac{\pi}{180}\sec(x^\circ+45^\circ)\tan(x^\circ+45^\circ)$

Ex. 4·10.

1. $\frac{3}{2}x^{\frac{1}{2}}$

2. $-\frac{1}{2}x^{-3/2}$

3. $-\frac{2}{3}x^{-5/3}$

4. $-3(x-1)^{-4}$

5. $\frac{9}{4}(3x+4)^{-\frac{1}{4}}$

6. $\frac{x+2}{2(x+1)^{3/2}}$

7. $\frac{-1}{x^2\sqrt{x^2+1}}$

8. $\dfrac{1-2x-x^2}{(1+x^2)^2}$ 9. $\dfrac{3x+2}{(3x^2+4x+1)^{3/2}}$

10. $\dfrac{2(x^2-2)}{\sqrt{(x^2-4)}}$ 11. $\dfrac{\cos 2x}{\sqrt{\sin 2x}}$

12. $\dfrac{1+\sin^2 x}{2\sqrt{\sin x}\cos^2 x}$ 13. $\dfrac{2(2x^2-1)}{\sqrt{1-x^2}}$

14. $\dfrac{1}{2\sqrt{x}}-1$ 15. $\dfrac{-a^2}{x^2\sqrt{a^2-x^2}}$

16. $\dfrac{-4}{(x^2-4)^{3/2}}$ 17. $\dfrac{x^3-2a^2x}{(x^2-a^2)^{3/2}}$

18. $\dfrac{k^2\tan x\sec^2 x}{\sqrt{1+k^2\tan^2 x}}$ 19. $\dfrac{b\tan x\sec^2 x}{\sqrt{a+b\tan^2 x}}$

20. $\dfrac{\cos\sqrt{x}}{2\sqrt{x}}$ 21. $\dfrac{\cos\sqrt{x}}{4\sqrt{x}\sin\sqrt{x}}$

22. $\dfrac{2x-(a+b)}{2\sqrt{(x-a)(x-b)}}$ 23. $\dfrac{1+\sin^2 x}{2\sqrt{\sin x}\cos^2 x}$

24. $\dfrac{\cos x\sin\sqrt{x}-\sqrt{x}\sin x\cos\sqrt{x}}{2\sqrt{x}\cos x\cos^2\sqrt{x}}$

25. $\dfrac{-1}{2\sqrt{x}(x-1)^{3/2}}$

26. $\dfrac{(aB-bA)x^2+(aC-cA)x+(bC-cB)}{(ax^2+2bx+c)(Ax^2+2Bx+C)^{3/2}}$

27. $\dfrac{1}{\sqrt{x-1}\,(x+1)^{3/2}}$

28. $\sin^{p-1}x\cos^{q-1}x\,[p\cos^2 x-q\sin^2 x]$

29. $\dfrac{1}{2\sqrt{x+1}}-\dfrac{1}{2\sqrt{x}}$ 30. $\dfrac{(a-b)\sin 2x}{2\sqrt{a\sin^2 x+b\cos^2}}$

Ex. 4·13.

1. $\dfrac{-1}{\sqrt{a^2-x^2}}$
2. $\dfrac{-a^2}{a^2+x^2}$
3. $\dfrac{2x}{\sqrt{1-x^4}}$
4. $\dfrac{1}{2\sqrt{x}\,(1+x)}$.
5. $\dfrac{1}{x\sqrt{4x^2-1}}$
6. $\dfrac{-1}{x\sqrt{x^2-1}}$
7. $\dfrac{1}{1+x^2}$
8. $\dfrac{1}{\sqrt{1-x^2}}$
9. $\dfrac{1}{1+x^2}$
10. $\dfrac{1}{2}$
11. $\dfrac{-2}{1+x^2}$
12. $\tan^{-1}x+\dfrac{x}{1+x^2}$
13. -1
14. -1
15. -1
16. $2x\tan^{-1}x$
17. $\dfrac{-x}{\sqrt{1-x^2}}\sin^{-1}x$
18. 0
19. 0
20. $\dfrac{2\sin\theta}{\sqrt{1-4\cos^2\theta}}$

Ex. 4·15.

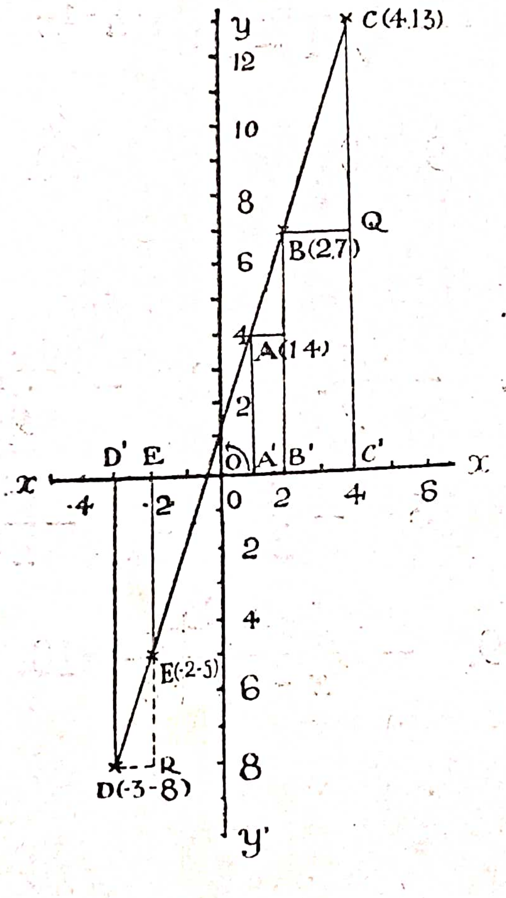
1. $-\dfrac{x}{y}$
2. $-\dfrac{2x}{3y}$.
3. $-\dfrac{9x}{4y}$
4. $-\dfrac{-b^2x}{a^2y}$
5. $\dfrac{b^2x}{a^2y}$
6. $-\dfrac{y}{2}$ or $-\dfrac{c^2}{x^2}$
7. -1
8. $\dfrac{y-2y}{14y-x}$
9. $-\dfrac{2x+y+4}{x-6y+3}$
10. $-\dfrac{ax+hy+g}{hx+by+f}$.

CHAPTER V

GEOMETRICAL MEANING OF DERIVATIVE

5·1 The slope of a straight line

We have drawn below the graph of the function $y=3x+1$. Here y is equal to a first degree expression in x and the graph of the function is therefore a straight line. So the function $3x+1$ is appropriately called a *linear function* of x.



Ordinates AA^1, BB^1, CC^1, DD^1 and EE^1 to meet the graph of the function are drawn at $x=1, 2, 4, -3$ and -2. The co-ordinates of the points on the graph where the ordinates meet them are also shown. To measure the increase in the value of y i. e. of the function $3x+1$ *as x increases*, perpendiculars AP, BQ, and DR (to meet EE^1

produced) are drawn. It is easy to note that the triangles ABP, BCQ and DER are all similar. ∠PAB = ∠QBC = ∠RDE = θ, the angle made by the st. line graph with *the positive direction of the x - axis.*

$\therefore$ $\tan\theta = \dfrac{BP}{AP} = \dfrac{CQ}{BQ} = \dfrac{ER}{DR} = 3$ where each of the ratios stands for

$$\frac{\text{change in the value of } y}{\text{increase in the value of } x}$$

This shows that whatever be the interval of x that we take, the ratio between the change in the value of y and the increase in the value of x is the same, being equal to $\tan\theta$.

This fixed ratio (equal to $\tan\theta$) is called the *slope* or the *gradient* of the st. line.

Definition:—The slope of a st. line is the tangent of the angle made by the st. line, *with the positive direction* of the x - axis.

* Note that the convention of measuring the angle with the positive direction of the x - axis should be strictly followed.

* In order to get the correct angle made by the st. line with the positive x-axis, i. e. the angle whose tangent is equal to the slope of the line, it is essential that the units on the x - and y - axes should be the same.

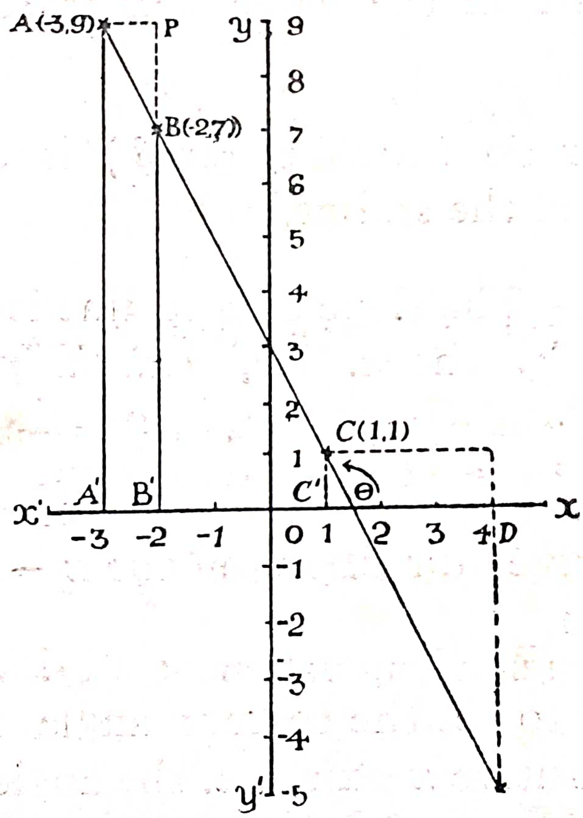
For example, we know that the slope of the line $y = x$ is unity i. e. tan 45°. Draw this line taking 1 in. = one unit both on the x - axis and on the y - axis. Then the line will make an angle of 45° with the positive x - axis. Draw the same line taking 1 in. = one unit on the x - axis and 2 in.=one unit on the $y-$axis. This line will make an angle of about 64° with the positive $x-$axis. Why? What will be this angle if the units are $\frac{1}{2}$ in.=one unit on the $x-$axis and 1 in.=one unit on the $y-$axis?

In the case of the st. line representing the function $y=3x+1$, the slope has a positive value, being equal to 3. But it need not always be so.

Take the following example:—

$$2x+y=3 \quad \text{i. e.} \quad y=-2x+3.$$

Here is the graph of the function.



Here again the triangles ABP; and CDQ are similar. $\angle PAB = \angle QCD = (180 - \phi)$ where ϕ is the angle between the st. line and *the positive direction of the x–axis*

$$\therefore \quad \tan \widehat{PAB} = \tan \widehat{QCD} = \tan(180 - \phi)$$

$$\text{i. e.} \quad \frac{PB}{AP} = \frac{DQ}{CQ} = \tan(180 - \phi)$$

i. e. $$\frac{2}{1} = \frac{6}{3} = \tan(180 - \phi)$$

But $$\tan(180 - \phi) = -\tan\phi$$

$$\therefore \quad \tan\phi = -2.$$

In fact $\tan\phi$ is defined as the slope of the st. line. So this st. line has a slope equal to -2.

The two cases above show that if $y = mx + c$, where m and c are constants, then m, *with its proper sign*, gives the slope of the line representing the function.

Any linear equation can be put into the above standard form.

Let $$2x - 3y + 4 = 0$$

Then $$y = \frac{2}{3}x + \frac{4}{3}$$

which shows that $\frac{2}{3}$ is the slope of the st. line representing the function. To show, in general, that if $y = mx + c$, then m is the slope of the line, is easy.

Take two values a and b of x where $b > a$.

The corresponding values of y are $ma + c$ and $mb + c$ respectively.

$$\text{Slope of the line} = \frac{\text{change in the value of } y}{\text{increase in the value of } x}$$

$$= \frac{(mb + c) - (ma + c)}{b - a}$$

$$= \frac{m(b - a)}{b - a}$$

$$= m.$$

* Differentiating $y = mx + c$, with respect to x, we get $\frac{dy}{dx} = m$. This also shows that the derivative of the function $y = mx + c$ w. r. t. x gives the slope of the graph of the function. In the case of this linear function, the derivative is a constant (m) and therefore the slope is also constant at all points i. e. for all values of x. This fact enables us to say that the graph of a function of the type $y = mx + c$, must be a st. line, for only a st. line has the same slope at all points on it. A graph which is not straight, but curved, does not have the same slope at all points, as will be shown later.

5·2. Exercise.

1. What are the slopes of the lines representing the following functions?

(i) $2y = 3x + 4$ (ii) $y + 3x = -1$

(iii) $\frac{x}{2} + \frac{y}{3} = 0$ (iv) $x = y$

(v) $y = 3$ (vi) $x = 2$.

Draw the graph in each case and say where it cuts the y–axis in each case.

2. What is the slope of the line $x \cos\alpha + y \sin\alpha = p$?

5·3 Tangents to a curve.

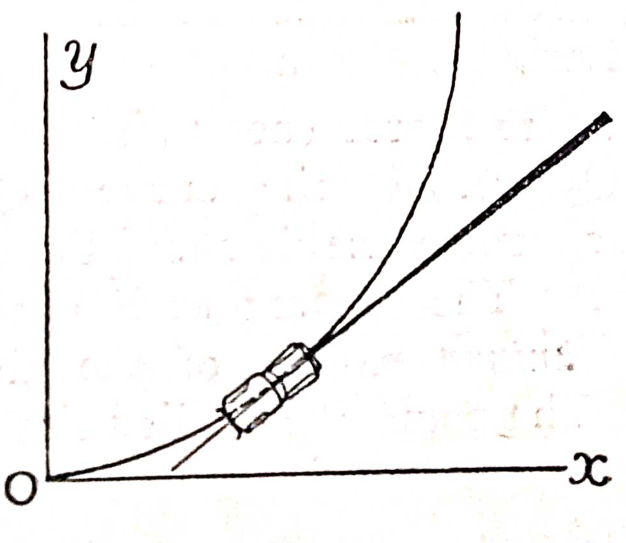
The tangent at a point P of a circle may be drawn accurately by constructing it at right angles to the radius through P. No such construction exists for drawing tangents to other curves.

Consider for example $10y = x^2$. Draw this curve from the given table of values taking 1 inch to represent 1 on each axis.

$x =$	-1	0	1	2	3	4	5	6	7
$y =$	0·1	0	0·1	0·4	0·9	1·4	2·5	3·6	4·9

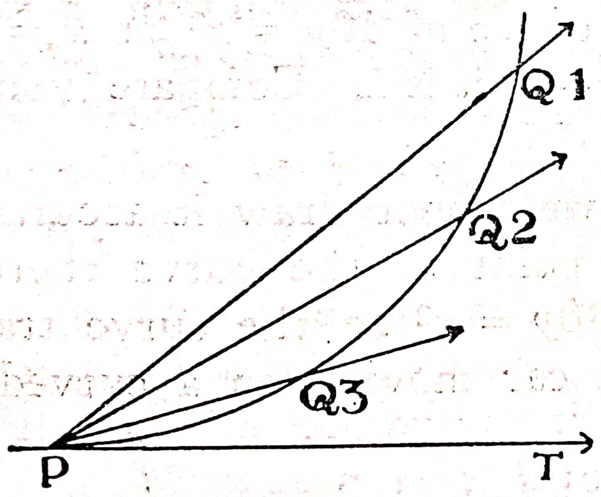
Draw as accurately as you can, judging by your eye the tangent at $x = 2$ and $x = 5$. Determine the gradients of these tangents. Compare your answers with those of other members of the class and note that all do not quite agree. This is rather unsatisfactory and therefore *we shall learn in this chapter how to calculate the gradient of the tangent at any point of a curve whose equation is given.* It will then be possible to show by calculation that the gradient or slope of the tangent to the curve of $10y = x^2$ at $x = 2$ is exactly 0·4 and that at $x = 5$ it is 1. Compare your answers with these figures.

Although we cannot draw it accurately, we assume that at every point of the curve there is a definite tangent. Let $10y = x^2$ be the curve traced out by the headlamp of a car moving on a curved road and let P



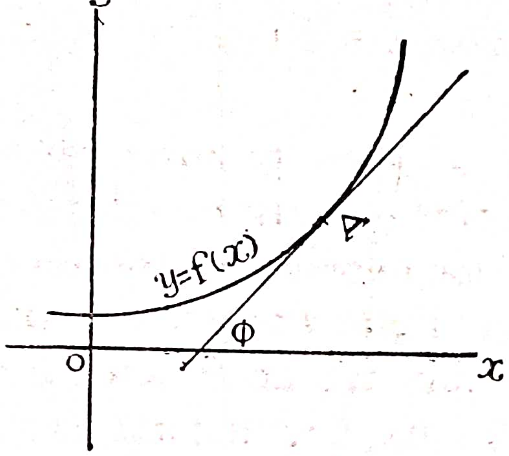
denote any position of the lamp. Then if the headlamp throws a narrow beam of light straight ahead of the car, this beam would lie along part of the tangent at P to the curve $10y = x^2$. The tangent maintains all along its length the instantaneous direction of the motion of the lamp at P.

Now consider a more geometrical way of looking at the tangent at a point of a curve. PQ is a curved railway line on which a locomotive is travelling from Q to P. Imagine that it is night and that a boy crouches in a hole dug between the rails at P with his eyes just above the ground level. As the locomotive approaches, he keeps a narrow horizontal beam of light focussed on the bottom of the front coupling. The beam PQ passes through the positions PQ_1, PQ_2 ... and settles down in the position PT as the locomotive thunders over the boy's head. PT is the tangent at P to the curve PQ.



PQ_1, PQ_2 ... are chords (not to be regarded as terminating at Q_1, Q_2 etc.) of the curve PQ. PT is the position in which the variable chord PQ settles down as Q approaches P. The tangent at P is then said to be defined as the *limiting position* of the chord PQ as Q approaches P. The point P is called the *point of contact* of the tangent.

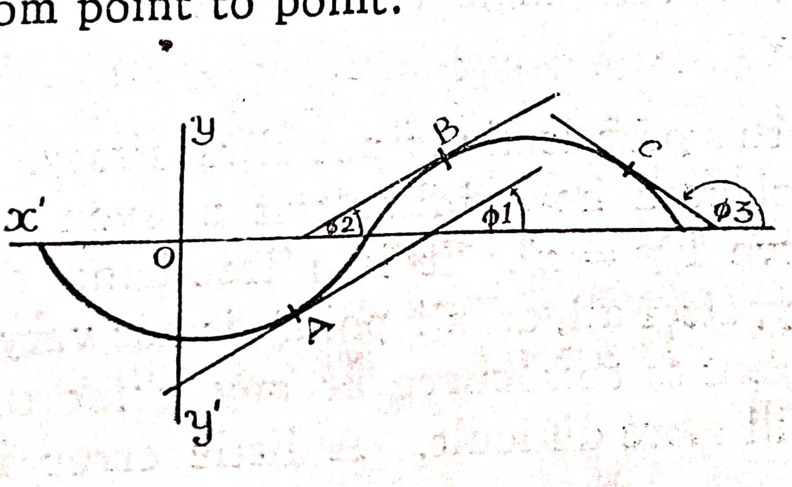
5·4 Definition of the slope of a curve at any point on it.



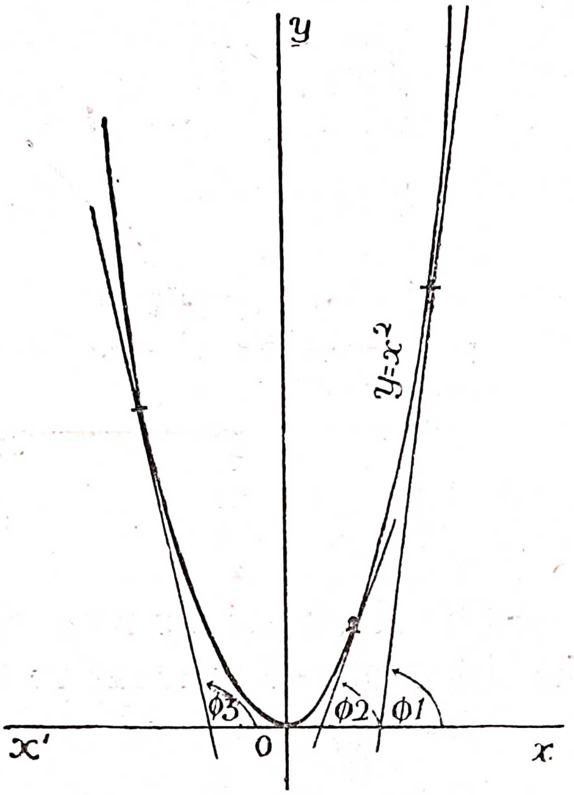
Definition:—The slope of a curve at any point on it is defined as the slope of the tangent to the curve at that point.

In the above figure A is any point on the curve of the function $y=f(x)$. The tangent to the curve at A meets the $x-$axis when produced and ϕ is the angle between this tangent and the positive direction of the $x-$axis. tan ϕ is the slope of the tangent at A. Therefore by definition tan ϕ is the slope of the curve *at A*.

With this definition, if the graph of a function is a curved line, the following figure will show the slope of the curve at different points of the curve is different. It varies from point to point.



We know the graph of $y=x^2$. It is drawn below to show that it has different slopes at different points on it.



5·5. The problem of finding the slope of a curve.

According to the definition given above, if we want to find the slope of a curve at any point on it, we have to first draw the curve from the given functional relation between the variables x and y. Then at any point on the curve that we want, we have to draw a tangent to the curve. We then measure the angle between this tangent and the positive direction of the x–axis. The value of the tangent of this angle is the slope of the curve at that point.

To do this practically is not always easy. You must have experienced how difficult it is even to draw the smooth curve $10y = x^2$. It is a free-hand drawing; so its accuracy, after all, at all points is not very high. To draw a tangent to this curve at any point on it, accurately, is still more difficult. A little error in drawing this line will result in some error in the angle measured. Consequently the value of the slope will also be inaccurate.

A theoretical method of knowing the slope of a curve at any point on it is therefore badly needed. Calculus gives us such a method.

5·6. The derivative as the slope of the curve or the geometrical interpretation of the derivative.

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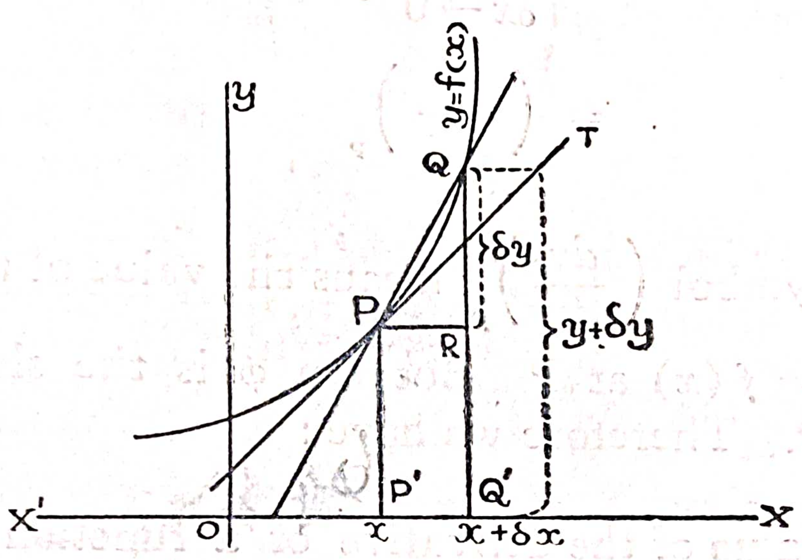


Consider the curve of any function $y = f(x)$. Take any point P (x, y) on the curve and another point Q near it whose co-ordinates are $(x + \delta x, y + \delta y)$ where δx is a small positive quantity i. e. a small increase in the value of x. Draw the ordinates PP' and QQ' to the curve. Draw PR ⊥ to QQ'. Join PQ and produce it to meet the x–axis. Let ϕ' be the angle between PQ and the positive direction of the x–axis.

Then in $\triangle$ PQR, $\angle QPR = \phi'$; PR $= \delta x$ and QR $= \delta y$.

$$\therefore \quad \tan \phi' = \tan \angle QPR = \frac{QR}{PR} = \frac{\delta y}{\delta x}$$

As Q approaches P, in the limit, PQ becomes the tangent PT to the curve at P. At the same time, i. e. as Q approaches P, in the limit $\delta x \to 0$.

and $\phi' \rightarrow \phi$, the angle between PT and the positive direction of the x-axis

$$\therefore \quad \tan \phi' \rightarrow \tan \phi$$

For this reason we can now say that

$$\therefore \quad \tan \phi = \lim_{Q \rightarrow P} \tan \phi'$$

$$= \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x}$$

$$= \left(\frac{dy}{dx} \right)_P$$

The symbol $\left(\frac{dy}{dx} \right)_P$ means the value of the derivative of $y = f(x)$ at P. But $\tan \phi$ is the slope of the curve at P. Therefore we have :

The value of the derivative of a function $y = f(x)$ for a particular value a of x, is the slope of the curve of the function at the point where $x = a$.

5·7. Illustrative examples :

1. Find the slope of the curve $y = x^2$ where $x = 3$.

Solution : $y = x^2$

Differentiating w. r. t. x

$$\therefore \quad \frac{dy}{dx} = 2x$$

$$\therefore \quad \left(\frac{dy}{dx} \right)_{x=2} = 2 \times 3 = 6$$

$\therefore$ The slope of the curve at $x = 3$ is 6.

2. What is the slope of the curve $xy=c$ at $x=\frac{1}{2}$?

Solution :— $xy = c$

$$\therefore \quad y = \frac{c}{x}$$

Differentiating w. r. t. x

$$\therefore \quad \left(\frac{dy}{dx}\right) = -\frac{c}{x^2}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x=\frac{1}{2}} = -\frac{c}{(\frac{1}{2})^2}$$

$$= -4c$$

$\therefore$ The slope of the curve at $x = \frac{1}{2}$ is $-4c$.

3. In the above example, at what point on the curve is the tangent to the curve perpendicular to the x-axis?

Answer :—If the tangent is perpendicular to the $x-$axis. Then the slope of the curve at that point is tan 90° i. e. infinity.

$$\text{Here } \frac{dy}{dx} = -\frac{c}{x^2}$$

$\therefore \quad \frac{dy}{dx}$ tends to infinity as $x \to 0$.

$\therefore$ The tangent to the curve at $x = 0$ is perpendicular to the x-axis.

4. In the curve of $y = \sin x$ at what points of the curve are the tangents to the curve (i) parallel to the $x-$axis (ii) inclined at 45° to the positive $x-$axis?

Solution :— $y = \sin x$

Differentiate w. r. t. x

$$\therefore \quad \frac{dy}{dx} = \cos x,$$

(i) At points where the tangents are parallel to the $x-$axis, the slope of the tangent, i. e. of the curve is zero.

$$\therefore \quad \frac{dy}{dx} = 0 \text{ if } \cos x = 0$$

But $\cos x = 0$ wherever $x = (2n + 1)\frac{\pi}{2}$ where n is any positive or negative integer.

$\therefore \quad \frac{dy}{dx} = 0$ at all points of the curve given by $x = (2n + 1)\frac{\pi}{2}$

$\therefore$ The tangent to the curve of $y = \sin x$ is parallel to the $x-$axis at all points on it given by $x=(2n + 1)\frac{\pi}{2}$ where n is any positive or negative integer.

(ii) Where the tangent to the curve is inclined at 45° to the positive $x-$axis, its slope is tan 45° i. e. 1.

$$\therefore \quad \frac{dy}{dx} = 1$$

i. e. $\cos x = 1$

$\therefore \quad x = 2n\pi$ where n is any positive or negative integer.

$\therefore$ At all points on the curve of $y = \sin x$ gixen by $x = 2n\pi$ where n is any positive or negative integer, the tangents to the curve make angles of 45° with the positive $x-$axis.

* Wherever $x = 2n\pi$, $y = \sin 2n\pi = 0$ which shows that at these points the curve cuts the $x-$axis.

(See the graph of this function drawn in Ch. I)

5. At what points of the curve $y = \frac{x-1}{x+1}$ is the slope of the curve equal to 2?

Solution—

$$y = \frac{x-1}{x+1}$$

$$\therefore \quad \frac{dy}{dx} = \frac{2}{(x-1)^2}$$

If $\frac{dy}{dx} = 2$, then $\frac{2}{(x-1)^2} = 2$

$$\therefore \quad (x-1)^2 = 1$$

$$\therefore \quad x-1 = \pm 1$$

$$\therefore \quad x = 0 \text{ or } 2.$$

∴ At the origin and at $x = 2$, the slope of the curve is equal to 2.

6. Determine the coefficients A, B and C so that the curve $y = Ax^2 + Bx + C$ may pass through the point $(-2, 8)$ and touch the axis of x at $(2, 0)$.

Solution:—$y = Ax^2 + Bx + C$ passes through $(-2, 8)$

$$\therefore \quad 8 = 4A - 2B + C \quad \ldots \quad \ldots \quad \text{I}$$

The curve touches the x–axis at $(2, 0)$.

∴ $(2, 0)$ lies on the curve.

$$\therefore \quad 0 = 4A + 2B + C \quad \ldots \quad \ldots \quad \text{II}$$

The axis of x touches the curve at $(2, 0)$

∴ the slope of the curve at $(2, 0)$ is the slope of the x–axis i. e. zero.

$$\therefore \quad \text{Slope of the curve} = \frac{dy}{dx}$$

(diff. w. r. t. x) $\quad = 2Ax + B$

Putting $x = 2$, slope of the curve at $(2, 0)$ is equal to $2A \times 2 + B$.

By data $\therefore \quad 0 = 4A + B \quad \ldots \quad \ldots \quad \text{III}$

To get the values of A, B and C solve the equations :—

$$8 = 4A - 2B + C \quad \ldots \quad \ldots \quad \text{I}$$
$$0 = 4A + 2B + C \quad \ldots \quad \ldots \quad \text{II}$$
$$0 = 4A + B \quad \ldots \quad \ldots \quad \text{III}$$

II—I gives $4B = -8$

$$\therefore \quad B = -2 \quad \ldots \quad \ldots \quad (\text{i})$$

Putting this value of B in III, we get

$$A = \tfrac{1}{2} \quad \ldots \quad \ldots \quad (\text{ii})$$

Putting the values of A and B in II we get $C = 2 \quad \ldots \quad \ldots \quad (\text{iii})$

$\therefore$ The function is $y = \frac{1}{2}x^2 - 2x + 2$.

5·8. Examples :—

1. Find the slopes of the following curves at the points mentioned against them.

(i)	$y = 1 - x^2$	$(-1, 0)$
(ii)	$y = 4x - x^2$	$(2, 4)$
(iii)	$y = x - x^3$	$(1, 0)$
(iv)	$y = x^3 - 5x^2 + 15$	$(3, -3)$
(v)	$y = \frac{1}{4}x^4$	$(-2, 4)$
(vi)	$y = (x^2 - 2)^2$	$(1, 1)$.

2. Find the slopes at $(1, 1)$ and $(4, 16)$ on the curve $y = x^2$. Find the average slope from $(1, 1)$ to $(4, 16)$.

3. Show that the gradient of $y = 4x^2$ at $(3, 36)$ is three times the gradient at $(1, 4)$.

4. At what point does the curve $y = 6x - x^2$ have a slope ~~of~~ -2?

5. At what points of the curve $y = x^3 - 3x^2 - 8x + 7$ are the tangents to the curve inclined at 45° to the positive x-axis?

6. Find the co-ordinates of the points on the curve $y = 32x^3 - 4x^2 - 8x + 2$ where the tangents are parallel to the axis of x.

7. At what points on the curve of $y = \sin x + \cos x$ do the tangents make an angle of 45° with the x-axis? Consider only the range 0 to 2π of x.

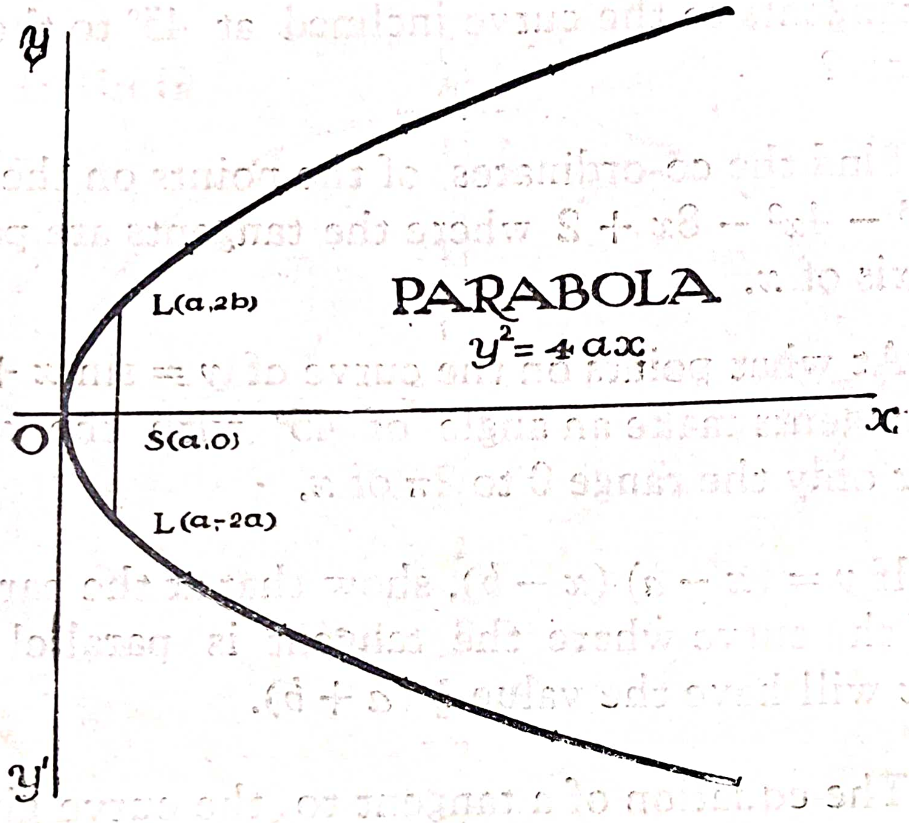
8. If $y = (x - a)(x - b)$, show that at the particular point of the curve where the tangent is parallel to the x-axis, x will have the value $\frac{1}{2}(a + b)$.

9. The equation of a tangent to the curve given by $y = 5 - 2x + 0{\cdot}5\,x^3$, being of the form $y = mx + c$ where m and c are constants, find the value of m and c if the point where the tangent touches the curve has an abscissa $x = 2$.

10. Determine the coefficients A, B, C and D so that the curve $y = \mathrm{A}x^3 + \mathrm{B}x^2 + \mathrm{C}x + \mathrm{D}$ may pass through the point $(-2, 8)$, touch the axis of x at the point $(2\ 0)$ and have its tangent parallel to the axis of x at the point for which $x = -1$. (Lamb.)

11. Find the slope of the curve $y^2 = 4x$ at the points whose x- co-ordinate is 2. At what point on the curve is the slope equal to unity?

12. The curve whose equation is of the type $y^2 = 4ax$, a being a constant, is called a **parabola.** Show that the point whose co-ordinates are $(at^2, 2at)$ lies on the curve for all values of t. What is the slope of the curve at the point $(at^2, 2at)$?



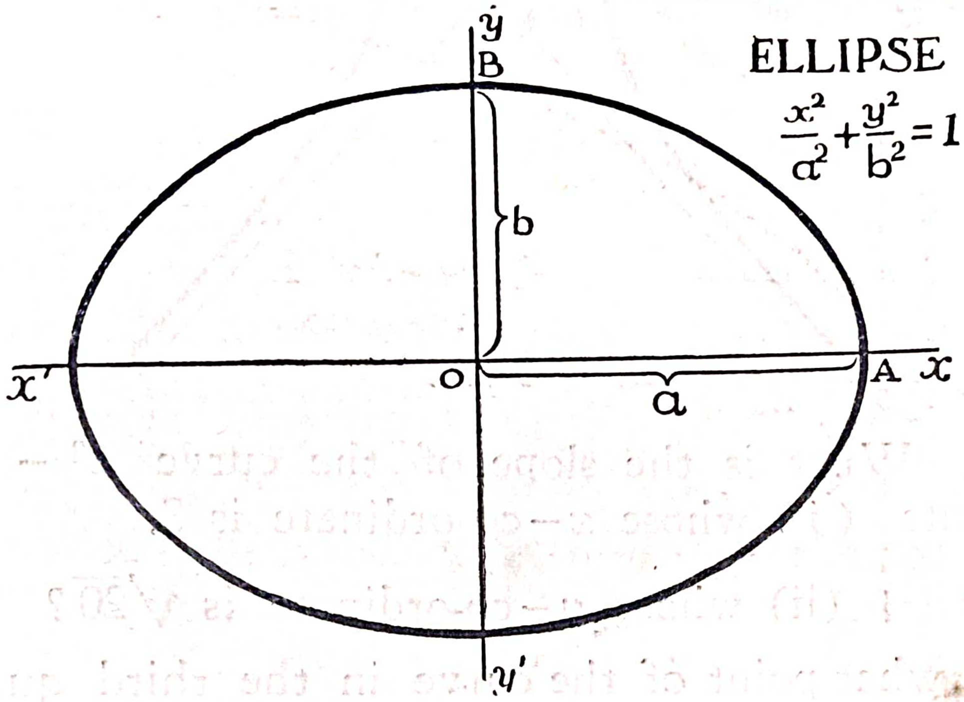
13. Show that the point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ always lies on the curve $y^2 = 4ax$ for all values of m. What is the slope of the curve at the point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$?

14. Find the slope of the curve $\frac{x^2}{4} + \frac{y^2}{9} = 1$ at the points (i) whose abscissa is equal to 1

(ii) whose ordinate is equal to 1.

At what points on the curve is the slope equal to 3?

15. The curve whose equation is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where a and b are constants, is called an **ellipse**. Show that a point whose co-ordinates are $(a \cos\theta\ b \sin\theta)$ lies on the curve, whatever be the value of θ. What is the slope of the curve at that point?

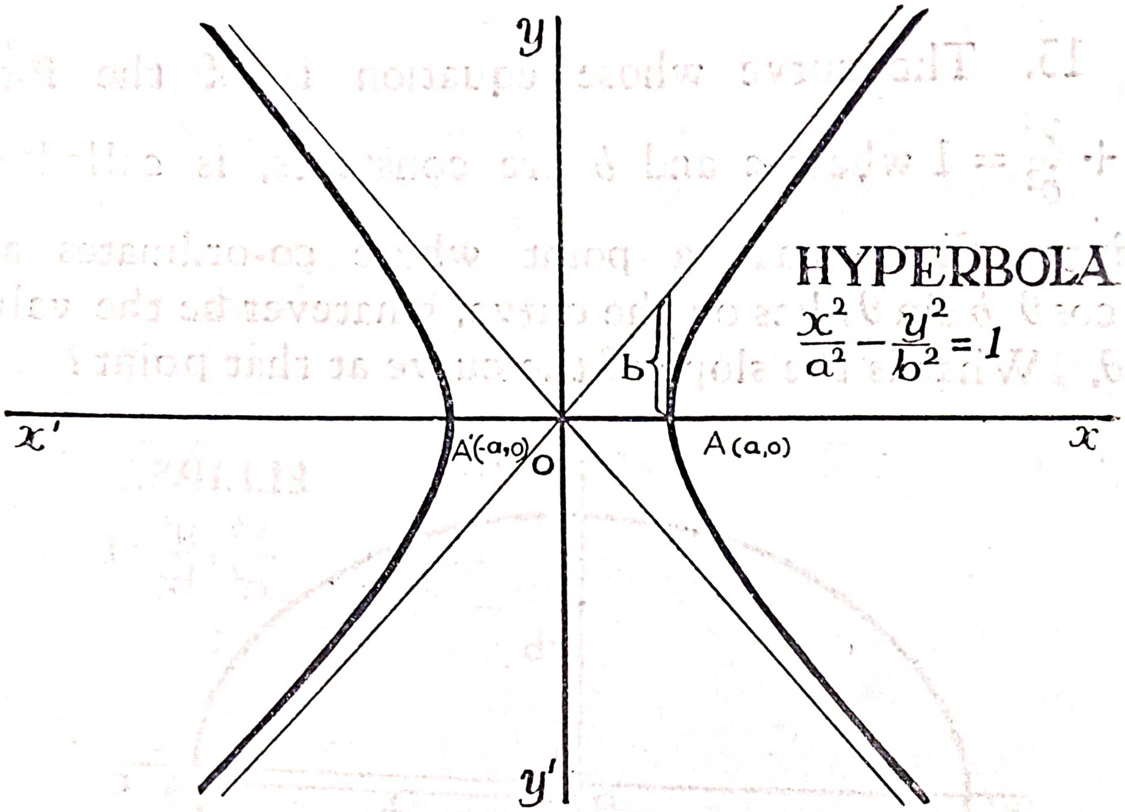


16. Find the slope of the curve $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at points (i) whose abscissa = 5.

(il) whose ordinate = −2.

At a point on the curve in the first quadrant, the tangent to the curve has a slope of 45°. Find the co-ordinates of that point.

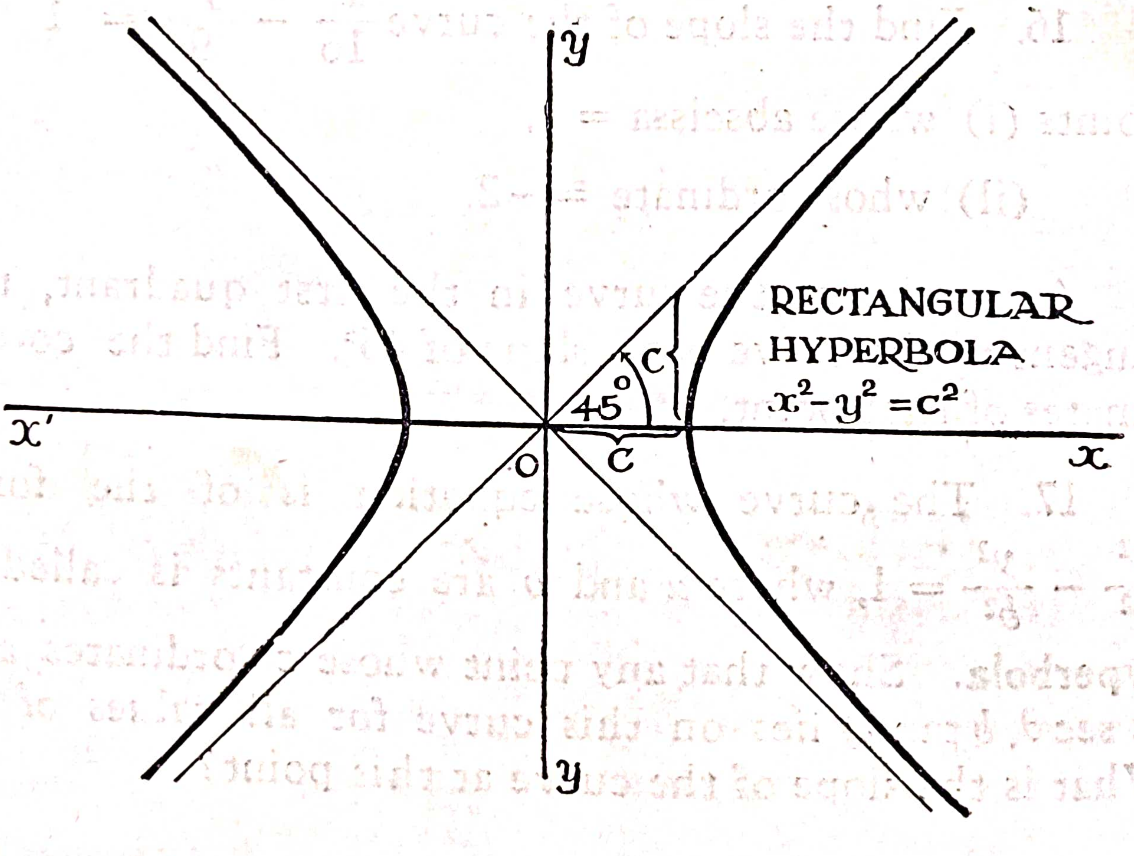
17. The curve whose equation is of the form $\frac{x^2}{y^2} - \frac{y^2}{b^2} = 1$, where a and b are constants is called a **hyperbola**. Show that any point whose co-ordinates are $(a \sec\theta, b \tan\theta)$ lies on this curve for all values of θ. What is the slope of the curve at this point?



18. What is the slope of the curve $x^2 - y^2 = 5$ at points (i) whose x-co-ordinate is 3.

(ii) whose y-co-ordinate is $\sqrt{20}$?

At what point of the curve in the third quadrant is the slope of the curve $= \frac{3}{2}$?



19. The curve whose equation is of the type $x^2 - y^2 = c^2$ where c is a constant is called a **Rectangular Hyperbola.** Show that any point whose co-ordinates are $(c \sec \theta, c \tan \theta)$ lies on this curve for all values of θ. What is the slope of the curve at this point? At what point on the curve is the slope equal to unity?

ANSWERS

Ex. 5·2.

1, (i) $\frac{3}{2}$. (ii) -3. (iii) $-\frac{3}{2}$. (iv) 1. (v) 0. (vi) infinity. The graphs cut the y-axis at the following points

(i) 2 (ii) -1 (iii) 0 (iv) 0 (v) 3 (vi) infinity.

2. $-\cot \alpha$.

Ex. 5·8.

1. (i) 2 (ii) 0 (iii) -2 (iv) -3 (v) -8 (vi) -4.
2. 2; 8; 5.
4. (4, 8).
5. (3, -17); (-1, 11).
6. $\left(-\frac{1}{4}, \frac{13}{4}\right)$; $\left(\frac{1}{3}, \frac{2}{27}\right)$.
7. $x = 0$ or $\frac{\pi}{2}$ or π or $\frac{3\pi}{2}$.
9. $m = 4$; $c = -3$.
10. A $= 1$; B $= -\frac{3}{2}$; $c = -6$: D $= 10$.
11. $\frac{1}{\sqrt{2}}$; (1, 2).

12. $\frac{1}{t}$.

13. m.

14. (i) $\pm \frac{\sqrt{3}}{2}$ (ii) $\pm 3\sqrt{2}$ (iii) $\left(\pm \frac{4}{\sqrt{5}}, \mp \frac{3}{\sqrt{5}}\right)$

15. $-\frac{b}{a}\cot\theta$.

16. (i) $\pm \frac{5}{4}$ (ii) $\pm \sqrt{3}$ (iii) $\left(\frac{16}{\sqrt{7}}, \frac{9}{\sqrt{7}}\right)$.

17. $\frac{b}{a \sin\theta}$.

18. (i) $\pm \frac{3}{2}$; (ii) $\pm \frac{\sqrt{5}}{2}$ (iii) $(-2, -3)$.

19. (i) $\operatorname{cosec}\theta$ (ii) point at infinity.

CHAPTER VI

PHYSICAL MEANING OF DERIVATIVES

6·1 The meaning of the term "rate"

Many of the important problems in calculus are those where *time* is the independent variable and we have to think about the values of some other quantity that varies as the time varies. Some things grow larger as time goes on; some other things grow smaller. The distance that a man travels from his starting place goes on ever increasing as time goes on. Boys grow taller as years go by.

The following is the schedule of timings of the Deccan Queen.

Miles from Bombay.			
...	Bombay V. T.	— Dep.	17·00 hrs.
62	Karjat Jn.	— Dep.	18·17 ,,
80	Lonavla	— Dep.	19·04 ,,
119	Poona Jn.	— Ar.	20·05 ,,

The three main stages of the journy are

(i) Bombay V. T. to Karjat Jn.
(ii) Karjat Jn. to Lonavla
(iii) Lonavla to Poona Jn.

We are interested to know which of the three stages is covered at the fastest rate and which of the stages at the slowest rate.

What do we mean by the word *rate* above? You might say, it is simple. It is the quotient of dividing the distance covered by the time taken to cover it; in other words, it is the distance covered in unit time. It is speed.

But the question, we should like to answer ourselves here is this. Does the *rate*, in the above case, depend either on the unit of distance or on the unit of time or on both, that we prefer to choose?

The answer is *NO*.

Take for example the case of a motor-car that flies past us going 11 yards per second. Calculation will show us that this is equivalent to a rate of 660 yards per minute or $22\frac{1}{2}$ miles per hour. Now in what sense is it true that a speed of 11 yards per second is the same as 660 yds. per minute or equal to $22\frac{1}{2}$ miles per hour? 11 yards is not the same as 660 yards nor is one second the same thing as one minute. Again 11 yards is entirely different from $22\frac{1}{2}$ miles; one second is also different from one hour.

What we mean by saying that the *rate* is the same is this:

The proportion borne between distance passed over and time taken to pass over it, is the same in the three cases.

In general, suppose a body moves from a particular point. Measure distances from that point to the position of the body at any time. Measure time also from the instant when the body starts from the initial position or the zero position. Denote the distance by the symbol s and the time by the symbol t.

$$s = 0 \text{ when } t = 0.$$

Let s be the distance covered in time t. Then $\frac{s}{t}$ gives the speed of the body during that period. If this speed is not uniform at all points, we shall have to specify that $\frac{s}{t}$ is the *average speed* during the period of time t. This average speed is ordinarily different from *the speed at any particular point* or *at any particular instant.*

If you want the velocity v at any time t, then take a small element of time δt during which the body moves a short distance δs. Then $v = \frac{\delta s}{\delta t}$. The smaller the element δt, the more accurate will be the value of v.

We shall shortly know how to find the accurate value of v, using differential calculus.

6·2. Rate of change (average rate of change).

Consider the function $y = x^2$ when $x = 3$, $y = 9$ and when $x = 5$, $y = 25$. For an increase of 2 in the value of x, the corresponding increase in the value of y is 16.

We define *rate of change* as

$$\frac{(\text{change in the value of the function})}{(\text{change in the value of the argument})}$$

In the above case the rate of change is equal to

$$\frac{25-9}{5-3} = \frac{16}{2} = 8.$$

Example 1. If $f(x) \equiv 3x^2 - 4x - 5$, find the rate of change in the value of $f(x)$ in the interval $(-1, 1)$ of x.

Solution:—$f(1) = 3\cdot1^2 - 4\cdot1 - 5$

$$= 3 - 4 - 5$$

$$= -6$$

$$f(-1) = 3\cdot(-1)^2 - 4(-1) - 5$$

$$= 3 + 4 - 5$$

$$= 2.$$

$$\therefore \quad \text{Rate of change} = \frac{-6-2}{1-(-1)}$$

$$= \frac{-8}{2}$$

$$= -4.$$

2. In the above example, calculate the rate of change in the value of the function between the values 1 and 2 of x.

Solution:—$f(2) = 3\cdot2^2 - 4\cdot2 - 5$

$$= 12 - 8 - 5$$

$$= -1$$

$f(1) = -6$ as already found out above.

$$\therefore \quad \text{Rate of change} = \frac{-1-(-6)}{2-1}$$

$$= \frac{-1+6}{1}$$

$$= 5$$

The two examples worked out above show that the rate of change of a function differs in the two intervals taken; which means that the rate of change differs from point to point. This can be also shown by taking smaller and smaller intervals for investigation of the variation of the rate of change.

When the rate of change of a function $f(x)$ varies at different points, then we define the rate of change expressed by the ratio

$$\frac{f(b)-f(a)}{b-a}$$

as the *average rate of change* of the function during the interval (a, b) of x.

6·3. The rate of change of a function at a point. (The derivative as a rate-measurer).

If the rate of change of a function varies from point to point, we are naturally interested in knowing what the rate of change of the function is *at a particular point* of the function.

Let us try to get an idea of the rate of change at a point by the following example:

$$\text{Let } y = x^2.$$

We want to know the rate of change of y w. r. t. x where $x = 3$.

Examine the following table :

Value of x	*Value of y*
2·9	8·41
2·99	8·9401
2·999	8·994001
3	**9**
3·001	9 006001
3·01	9·0601
3·1	9·61

Interval of x	*Rate of change of y*
(3, 3·1)	$\dfrac{9.61 - 9}{3.1 - 3} = 6.1$
(3, 3·01)	$\dfrac{9.0601 - 9}{3.01 - 3} = 6.01$
(3, 3·001)	$\dfrac{9.006001 - 9}{3.001 - 3} = 6.001 \downarrow$
Limiting value	$\to 6$
(3, 2·999)	$\dfrac{9 - 8.994001}{3 - 2.999} = 5.999 \uparrow$
(3, 2·99)	$\dfrac{9 - 8.9401}{3 - 2.99} = 5.99$
(3, 2·9)	$\dfrac{9 - 8.41}{3 - 2.9} = 5.9$

The table shows that when you find the average rate of change in the interval $(3 - h, 3)$ or $(3, 3 + h)$ and make h smaller and smaller, there is a definite tendency for the rate to approach the value 6. So we can say that when $x \to 3$, the rate of change of y w. r. t. x tends to 6. This we can put as follows:

The rate of change of y w. r. t. x at $x = 3$ where $y = f(x)$ is defined as

$$\lim_{h \to 0} \frac{f(3 + h) - f(3)}{(3 + h) - 3}$$

$$\text{i. e.} \quad \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}.$$

In general the rate of change of $f(x)$ w. r. t. x at $x = a$ can be defined as

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

More generally, replacing a by x and h by δx, we have the rate of change of $f(x)$ w.r.t. x as

$$\lim_{\delta \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

But the above limit is nothing but the differential coefficient of $f(x)$. So we say that **the derivative of a function is also a rate-measurer.**

It follows then that if we want to calculate the rate of change of $f(x)$ at $x = a$, then we have to find out the derivative $f'(x)$. Then $f'(a)$ gives us the rate of change of $f(x)$ at $x = a$.

In the above example $f(x) = x^2$

$\therefore$ $f'(x) = 2x$ and where $x = 3$, its value $= 2 \times 3 = 6$ which is the rate of change of x^2 with x where $x = 3$.

6·4. Examples

1. For what value of x does the function $x - 2x^{-1}$ have a rate of change of 3 with respect to x?

2 If $y = 6 - x^2$, find the rate of increase of y with x when $x = 2$. As x increases through the value 2, is y increasing or decreasing?

3. If $y = \frac{x^2}{10}$ find the rate of increase of y with x when $x = 5$.

4. You are given two formulæ,

$$y = \frac{x^2}{2} \text{ and } z = 4x - 8.$$

(i) Find the rate of increase of y with x when $x = 4$.

(ii) Find the constant rate of increase of z with x.

(iii) Show that $y=z$ when $x=4$.

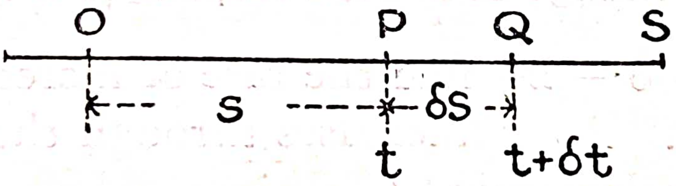
(iv) Sketch the graphs of the formulae on the same axes.

(v) What is the relationship of the straight line to the curve?

5. The force P lb. wt. on a plane surface of area 1 sq. ft. held perpendicular to a current of water flowing v ft./sec is given by the formula $P = 1{\cdot}8\,v^2$. What is the average rate of increase of force with the velocity of the current from $v = 2$ to $v = 5$? What is the rate of increase of P with v when $v=3$?

6·5. Rates: application to mechanics—velocity and acceleration at a point.

Suppose that a particle moves at a variable rate along the line OS and passes through the points P and Q at times t and $t + \delta t$, respectively. Choose as an origin a fixed point (say O) on the line. Let the displacements,



OP and PQ be s and δs respectively (the increment δs is considered positive or negative according as it lies to the right or to the left of P).

$$\text{Then } \frac{\delta s}{\delta t} = \text{average velocity during the interval } \delta t.$$

The velocity varies from point to point and we want the instantaneous velocity at P. This can be got by making $\delta t \to 0$ and taking the limit of $\frac{\delta s}{\delta t}$ as $\delta t \to 0$.

$$\therefore \lim_{\delta t \to 0} \frac{\delta s}{\delta t} = \frac{ds}{dt}$$

= velocity of the particle at the instant t,

or $\frac{ds}{dt}$ = velocity of the particle at the point P.

Let us denote the velocity $\frac{ds}{dt}$ at time t by the symbol v.

$$\therefore \quad v = \frac{ds}{dt}.$$

Let the velocity at $t + \delta t$ be $v + \delta v$. (δv may be positive or negative according as the velocity is increasing or decreasing.)

Then $\frac{\delta v}{\delta t}$ = average acceleration during the interval δt.

As before taking the limit as $\delta t \to 0$ we get

$$\lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt} \text{ acceleration at time } t.$$

or $\frac{dv}{dt}$ = acceleration at point P.

Since $v = \frac{ds}{dt}$, we have

acceleration $f = \frac{dv}{dt}$ where v is a function of s.

Therefore f is a function of v where v is a function of s. i. e. f is a function of a function.

$$\therefore \quad f = \frac{dv}{dt}$$

$$= \frac{dv}{ds} \cdot \frac{ds}{dt}$$

$$= \frac{dv}{ds} \cdot v$$

$\therefore\ f = v\frac{dv}{ds}$ which is another formula for the acceleration.

6·6. Illustrative Examples

1. If the distance-time formula is $s = t^2 - 2t$, show that the body is travelling 3 times as fast when $t = 4$ as when $t = 0$ and in the opposite direction.

Solution.— $s = t^2 - 2t.$

$$\therefore\ v = \frac{ds}{dt} = 2t - 2.$$

$$\therefore\ v_0 = -2 \text{ (putting } t = 0 \text{ above)}$$

$$\therefore\ v_4 = +6 \text{ (putting } t = 4 \text{ above)}$$

which shows that the body is travelling 3 times as fast ($v = +6$) when $t = 4$ as when $t = 0$ ($v = -2$) and in the opposite direction. ($\because$ The two velocities have opposite signs.)

2. If $s = t^3 - 6t^2 + 12t$ find the acceleration when the speed is zero.

Solution:— $s = t^3 - 6t^2 + 12t.$

$$\therefore\ v = \frac{ds}{dt} = 3t^2 - 12t + 12.$$

$$\therefore\ \text{Acceleration} = \frac{dv}{dt} = 6t - 12.$$

Speed is zero when $3t^2 - 12t + 12 = 0$

i. e. $t^2 - 4t + 4 = 0$

i. e. $t = 2.$

Acceleration when $t = 2$ is $6 \times 2 - 12$ i. e. zero.

$\therefore$ The acceleration is zero when the speed is zero.

3. Two particles, starting from the same point at the same time and moving on the same straight line, are at time t at distances

$$s_1 = t^2 - 9t, \quad s_2 = 3t - 2t^2.$$

from the starting point. When will the particles have the same speed? What are their speeds when they have the same position?

Solution:—

$$s_1 = t^2 - 9t$$

$$\therefore \quad v_1 = \frac{ds_1}{dt} = 2t - 9 \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

$$s_2 = 3t - 2t^2$$

$$\therefore \quad v_2 = \frac{ds_2}{dt} = 3 - 4t$$

$$\therefore \quad v_1 = v_2 \text{ if } 2t - 9 = 3 - 4t.$$

$$\text{i. e. } 6t = 12$$

$$\text{i. e. } t = 2$$

$\therefore$ The two particles have the same speed when $t = 2$.

The two particles have the same position when $s_1 = s_2$.

$$\text{i. e. } t^2 - 9t = 3t - 2t^2$$

$$\text{i. e. } 3t^2 - 12t = 0$$

$$\text{i. e. } t(t - 4) = 0$$

$$\text{i. e. } t = 0 \text{ or } t = 4$$

Case (i) when $t = 0$ i. e. the starting point

$$v_1 = 2t - 9$$

$$= -9 \quad (\text{putting } t = 0)$$

$$v_2 = 3 - 4t$$
$$= 3 \quad \text{(putting } t = 0)$$

Case (ii) when $t = 4$.

$$v_1 = 2t - 9$$
$$= -1 \quad \text{(putting } t = 4)$$
$$v_2 = 3 - 4t$$
$$= -13 \text{ (putting } t = 4)$$

4. A particle moves in a straight line and its distance from a fixed point at time t is given by the equation $s = a \cos(\omega t)$

Find (i) its velocity at time t

(ii) its maximum and minimum velocities.

Prove that its acceleration varies as its distance from the fixed point, and is towards that point.

Solution:— $s = a \cos(\omega t)$

$$\therefore \quad v = \frac{ds}{dt} = -a\omega \sin(\omega t.).$$

a and ω being constants, the velocity is a maximum when numerically $\sin(\omega t)$ is a maximum,

i. e. when $\omega t = (2n + 1)\frac{\pi}{2}$ where n is an integer.

i. e. when $t = (2n + 1)\frac{\pi}{2\omega}$.

This shows that the velocity is a maximum after every period of $\frac{\pi}{\omega}$

The velocity is a minimum when

$$\sin \omega t = 0$$

i. e. $\omega t = n\pi$ where n is an integer.

i. e. $t = \frac{n}{\omega}\pi$.

This shows that the velocity is a minimum after every period of $\frac{\pi}{\omega}$

$$\therefore \quad \text{Max. velocity} = -aw \sin (2n + 1)\frac{\pi}{2}$$

$$= \pm a\omega$$

$$\text{Min velocity} = -a\omega \sin n\pi$$

$$= 0.$$

The motion of the particle is therefore *periodic*.

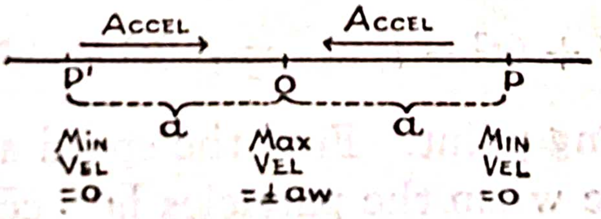
$$\text{Again acceleration} = \frac{dv}{dt}$$

$$= -a\omega^2 \cos(\omega t)$$

$$= -\omega^2 s.$$

$\therefore$ The acceleration is proportional to displacement. Again if s is positive the acceleration is negative and if s is negative the acceleration is positive. It is thus seen that the acceleration is always directed towards the origin, i. e. the point from which s is measured.

See the following figure



The particle swings forwards and backwards between P and P' with an *amplitude a*.

Such a motion is called *Simple Harmonic Motion*.

6·7. Exercises.

Find the speed and acceleration:

1. $s = 4t + 8$
2. $s = 5 - 2t$
3. $s = 12t - 3t^2$ when $t = 2$
4. $s = 16t - 4t^2$ when $t = 1$
5. $s = 8 + 64t - 16t^2$ when $t = 3$
6. $s = 9 - 64t + 8t^2$ when $t = 2$
7. $s = 9t - t^3$ when $t = 0$
8. $s = 10 - 18t^2 + t^3$ when $t = 1$
9. $s = 15 - 27t + 9t^2 - t^3$ when $t = 0{\cdot}5$
10. $s = t^4 - 2t^2 + 1$ when $t = 5$

11. If $s = t^2 - 6t + 4$, find the value of s when the speed is zero.

12. If $s = 1 + 6t^2 - t^3$, when will the speed be 9 ft./sec.?

13. $s = t^3 - 6t^2 + 8t - 4$ when is the acceleration 12 ft./sec.^{2}? What is the speed at this instant?

14. Two particles, starting from the same point at the same time and moving on the same straight line, are at time t at distances

$$s_1 = t^3 + 6t^2 - t, \qquad s_2 = 3t^3 - 4t^2 + 7t$$

from the starting point. Find the speed and acceleration of each particle when the particles last coincide.

15. A body moves in such a way that the spaces described in the time t from starting is given by $s = t^n$ where n is a constant. Find the value of n when the velocity is doubled from the 5th. to the 10th. second;

find it also when the velocity is numerically equal to the acceleration at the end of the 10th. second.

16. A point moves in a straight line such that $v^2 = k(a^2 - s^2)$. Show that the acceleration is always pointed towards the orgin and is proportional to s.

17. A particle moves in a straight line and its distance (in feet) from a fixed point at time t secs. is given by the equation $s = 4 + 10t - t^2$.

Find (i) its velocity when it has been in motion for 2 secs.

(ii) its distance from the fixed point when its velocity is zero.

(iii) its acceleration.

Describe the motion of the particle.

18. A stone is thrown vertically upward from the top of a building 96 ft. high. Its height above the ground at any time is given by $s = 96 + 80t - 16t^2$.

Find (i) the initial speed ;

(ii) the time required to reach the highest point ;

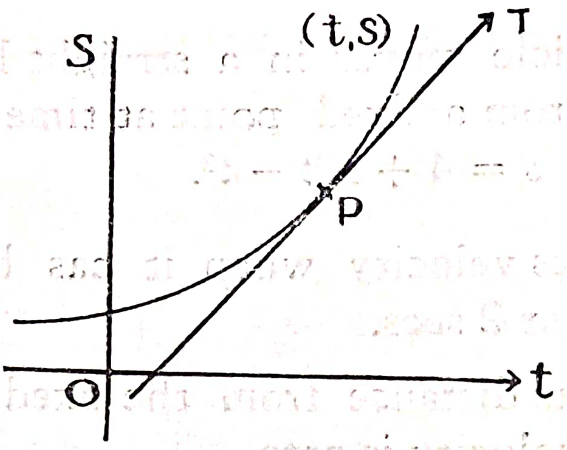
(iii) the maximum height above the top of the building reached by the stone ;

(iv) its speed on striking of the ground.

6·8. Space-time curve : velocity time curve :

Let a point, moving in a straight line be at a distance s from a fixed point O at an instant t, and let v, be the velocity at that instant. Then s and v are different for different instants of time.

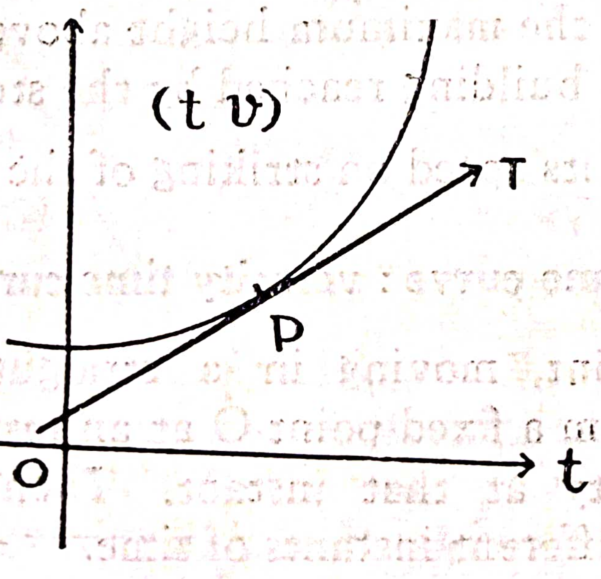
If (t_1, s_1), (t_2, s_2), (t_3, s_3) ...... are sets of values of t and s, and taking t as the abscissa and s as the ordinate, if we plot the points (t_1, s_1), (t_2, s_2), (t_3, s_3)..., the resulting graph is called the *space-time curve*, or (s, t) curve.



The slope of this curve at any point P is the slope of the tangent to it at P i. e. $\frac{ds}{dt}$ or v.

∴ *The slope of the tangent at any point of the space-time curve* $= \frac{ds}{dt} =$ *velocity at the point.*

Similarly taking t along the abscissa and v along the ordinate we can draw a velocity-time curve



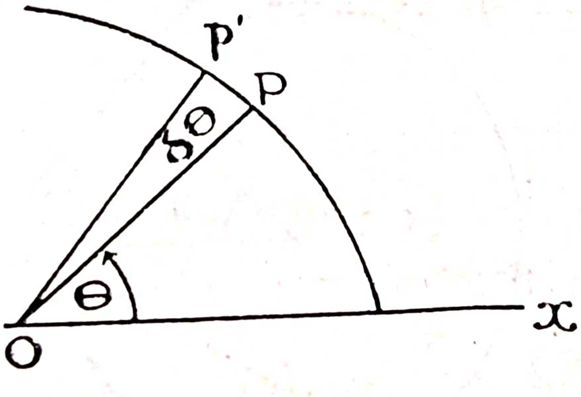
the slope of this curve at any point P on it is the slope of the tangent to it at that point i. e. $\frac{dv}{dt}$ or f, the acceleration.

$\therefore$ *The slope of the tangent at any point of a velocity-time curve* $= \frac{dv}{dt} =$ *acceleration at that point.*

Exercise: If $10s = t^3$, draw the space-time curve and from the curve read the velocity when $t = 3$. Also draw the velocity-time curve and from this curve read the acceleration when $t = 3$ Verify by calculation in each case.

* As explained in § 4·1 it is essential to take the same units along the two axes when the two graphs required in the above problem are drawn.

6·9. Angular velocity and angular acceleration (of a moving point P about a fixed point O).



Draw any fixed line OX. Join OP. Let $\angle POX = \theta$ measured in radians, where P is the position of the moving point at the end of time t. Let P' be the position of this point at the instant $t + \delta t$ where $\angle P'OX = \theta + \delta\theta$.

$$\therefore \quad \frac{\delta\theta}{\delta t} \text{ is the rate of change of } \theta \text{ w. r. t. time.}$$

$$\therefore \quad \lim_{\delta t \to} \frac{\delta\theta}{\delta t} = \frac{d\theta}{dt}$$ is the *angular velocity* of P about the fixed point O.

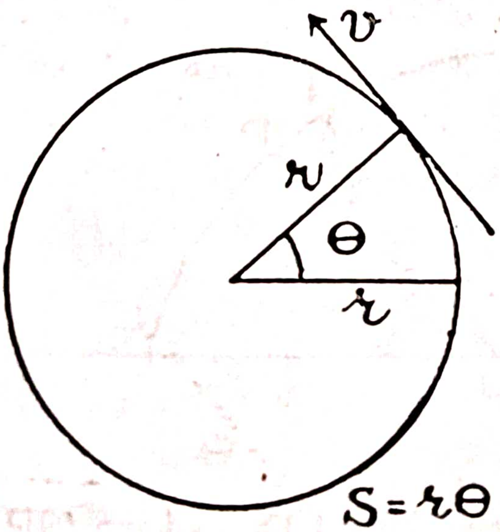
We use the symbol ω (omega) to denote angular velocity.

$$\therefore \quad \text{Angular velocity } \omega = \frac{d\theta}{dt}.$$

Again if $\omega + \delta\omega$ be the angular velocity at time $t + \delta t$, arguing as before we get the *angular acceleration* of P about O to be $\dfrac{d\omega}{dt}$.

6·10. Examples :

1. A particle P moves on a circle of radius r and centre O. Show that the speed of P at any instant is $v = r\omega$ where ω is the angular velocity of OP in radians per unit time.



[Hint : Let $s = r\theta$ be the arc corresponding to the angular displacement of θ of OP].

2. Show that the tangential acceleration of a particle moving on a circle of radius r is, at any time, equal to $r\alpha$ where α is the instantaneous value of the angular acceleration of the radius to the particle in radians / (unit time $)^2$.

3. The minute hand of a clock is 8 in. long. Find the speed of its outer end.

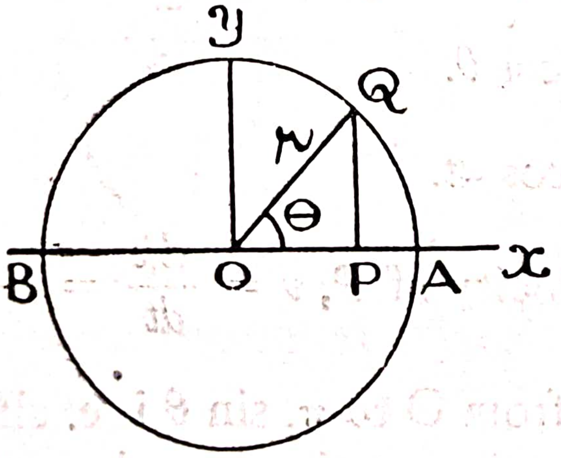
4. A boy runs round a circular track 200 yards long at 10 m. p. h. What is his angular velocity about the centre of the track ?

5. A point moves in a circle with constant speed. Prove that its angular velocity about a point on the circumference is half that about the centre.

6. If $\theta = \sin 3t$, find θ and ω when the angular acceleration is -9 radians / sec^2.

6·11 Simple Harmonic Motion

Let Q be a point which moves in the counter-clockwise direction along the circumference of a circle of radius r with a constant speed, so that the radius OQ revolves about O, with constant angular velocity. Then the point P, the projection of Q on the x – axis will oscillate between A and B, describing what is known as *Simple Harmonic Motion* with O as *centre* and r as *amplitude.*



Suppose that the point Q (and hence the point P also) is situated at A. Then at the time t the radius OQ will have swept through an angle θ given by the equation

$$\theta = \omega t$$

where ω, the constant angular velocity of OQ will be expressed in radians per unit time.

When OQ sweeps through one complete circuit i. e. an angle of 2π radians, the point P completes one complete oscillation i. e. it moves from A to B and back to A again.

Denote by T, the time for one complete oscillation of P.

$$\therefore \quad \mathrm{T} = \frac{2\pi}{\omega}$$

being the same as the time required for OQ to revolve through an angle of 2π (i. e. one complete revolution) at the constant angular velocity ω.

The time T is called the *period* of the simple harmonic motion; its reciprocal, representing the number of periods per unit time, is called the *frequency* and is denoted by γ (gamma)

$$\therefore \quad \gamma = \frac{\omega}{2\pi}.$$

Let OP $= x$, the displacement of P at time t.

$$\therefore \quad x = r\cos\theta.$$

i. e. $x = r\cos\omega t.$

$$\therefore \quad \text{The velocity of P, } v = \frac{dx}{dt} = -r\omega\sin\omega t.$$

As θ varies from O to π, $\sin\theta$ i. e. $\sin\omega t$ is positive. $\therefore v = -r\omega\sin\ \omega t$ is negative i. e. the velocity of P is in the direction of negative x – axis or from A towards B. So during this interval P moves from A to B.

As θ varies from π to 2π, $\sin\theta$ i. e. $\sin\omega t$ is negative.

$\therefore$ The velocity of P $= -r\omega\sin\omega t$ is positive.

i. e. during this interval P moves from B back to A.

Again $v = -r\omega\sin\omega t$ is numerically the maximum when $\sin\omega t$ is numerically a maximum i. e. when $\omega t = \theta$ is either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ and therefore $\sin\omega t = \pm 1$.

$\therefore$ Maximum velocity $= \mp r\omega$.

This shows that P has the maximum velocity $-r\omega$ (in the negative direction of the x – axis) when $\theta = \frac{\pi}{2}$ i. e. when P passes through O while moving from A towards B.

Again P is the maximum velocity $+r\omega$ (in the positive direction of the x–axis) when $\theta = \frac{3\pi}{2}$ i. e. when P passes through O while it moves from B towards A.

Acceleration $\propto$ (alpha) of P.

$$= \frac{dv}{dt}$$

$$= -r\omega^2\cos\omega t$$

$$\therefore \quad \propto = -\boldsymbol{w^2x}.$$

Thus, for a particle describing simple harmonic motion, the acceleration is proportional to the displacement of the particle from a fixed point (here, the point O). This acceleration is always directed towards the fixed point, because when x is positive, $-\omega^2 x$ is negative and vice versa.

Therefore any motion for which the acceleration = minus a constant times displacement ($\alpha = -kx$) can be shown to be a simple harmonic motion.

$\therefore$ $\alpha = -kx$ where k is a positive constant is the characteristic equation of simple harmonic motion.

* The student will find the above information useful when he studies the motion of a pendulum.

Answers

Ex. 6·4.

1. ± 1. 2. -4; decreasing. 3. 1.
4. (i) 4. (ii) 4. (v) tangent at $x = 4$.
5. 12·6 ; 10·8.

Ex. 6·7.

1. 4; 0. 2. -2; 0. 3. 0; -6.
4. 8; -8. 5. -32; -32. 6. -32; 16.
7. 9; 0. 8. -33; -30. 9. $-18{\cdot}75$; 15.
10. 480; 296. 11. -5. 12. $t = 1$ or 3.
13. $t = 4$; $v = 8$.
14. $v_1 = 95$; $f_1 = 36$; $v_2 = 119$; $f_2 = 64$.
15. $n = 2$; $n = 11$. 17. (i) 6 ft./sec; (ii) 29 ft. (iii) -2 ft./sec.
18. (i) 80 ft./sec. (ii) $\frac{5}{2}$ secs.
(iii) 180 ft. (iv) 112 ft./sec.

Ex. 6·10.

3. 0·014 in./sec. 4. 0·31 radians/sec.
6. $\theta =$ one radian; $\omega = 0$.

CHAPTER VII

INTEGRATION

7·1. The nature of the problem:

So far we dealt with the process of finding the derivative of a given function. Now let us put to ourselves the converse problem.

"Given a function $f(x)$ to find a function $\phi(x)$ such that the derivative of $\phi(x)$ is equal to the given function $f(x)$.

For example if $f(x) = 2x$, then $\phi(x) = x^2$, because the derivative of x^2 is $2x$. Again if $f(x) = 3x^2$, then $\phi(x) = x^3$; if $f(x) = \cos x$, then $\phi(x) = \sin x$ and so on.

This is the problem of Integral Calculus. **Integration is the inverse process of differentiation.** The function we get thereby is called the **Integral** of the given function.

* $\int \ldots\ldots\ldots\ldots\, dx$ *is a symbol of operation meaning "integral of …… with respect to x."* We put the function to be integrated in the dotted space.

The integral of $2x$ w. r. t. x is x^2

i. e. $\int 2x\, dx = x^2$

The integral of $\cos x$ w. r. t. x is $\sin x$.

i. e. $\int \cos x\, dx = \sin x$.

7·2. The Indefinite Integral: the arbitrary constant.

We have seen in the previous chapters that given a function, its derivative was fixed and therefore unique. Not so in the inverse process of integration.

For example, consider again the functions $2x$ and x^2. By definition, "x^2 is the integral of $2x$"; but x^2 is not the only possible integral, because $x^2 - 1$, $x^2 + 2$, $x^2 + 3$ etc., in general, $x^2 + c$ where c is any *arbitrary constant*, all these have the same derivative $2x$. Therefore the integral of $2x$ is, strictly speaking $x^2 + c$, where c is any arbitrary constant.

We put this as $\int 2x \; dx = x^2 + c.$

Similarly $\int 3x^2 \, dx = x^3 + c.$

and $\int \cos x \, dx = \sin x + c$

and so on.

Treating integration as the inverse process of differentiation, we thus find that the integral of a function is *not definite*; for the arbitrary constant can be any quantity. We therefore call these integrals as *indefinite integrals*.

* Knowing full well that the arbitrary constant is always there, we do not think it necessary to put it down every time, *unless it is necessary for some other purpose*, and therefore we omit it and write simply as follows:—

$$\int 2x \; dx = x^2$$

$$\int 3x^2 \, dx = x^3$$

$$\int \cos x \, dx = \sin x$$

and so on.

Definition:—If $f(x)$ is the derivative of $\phi(x)$, then we call $\phi(x)$ as the *integral* of $f(x)$ with respect to x or the *primitive* of $f(x)$.

Symbolically:—

$$\text{If } \frac{d}{dx}\left[\phi(x)\right] = f(x),$$

$$\text{Then } \int f(x)\, dx = \phi(x)$$

The function $f(x)$ to be integrated is called the *integrand.*

With the above definition, we put $\int f(x)\, dx = \phi(x) + c$, where c is an arbitrary constant, which is omitted in the indefinite integral. We retain it in an integral only if it is required for any specific purpose.

* *Note*:—The derivative of $x + c$, where c is any constant is 1. Therefore the integral of 1 is $x + c$, or simply x, the indefinite integral. This we put as

$$\int dx = x + c \text{ or simply } \int dx = x.$$

Here we have to understand that if the gap in the operation $\int \ldots\ldots dx$ is not filled by any function of x, then it is to be understood that the number 1 stands there. So $\int dx$ means $\int 1 \cdot dx$ and is therefore equal to x.

7·3. Standard Forms:

Since integration is the inverse of differentiation, the results of integration of a few simple functions can be obtained at once from known standard forms of derivatives. They are given below for reference while solving examples on integration that follow:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \text{ except when } n = -1 \qquad (1)$$

$$\int (ax+b)^n\,dx = \frac{1}{a}\cdot\frac{(ax+b)^{n+1}}{n+1} \text{ except when } n = -1 \qquad (2)$$

$$\int \cos x\,dx = \sin x \qquad (3)$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) \qquad (4)$$

$$\int \sin x\,dx = -\cos x \qquad (5)$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) \qquad (6)$$

$$\int \sec^2 x\,dx = \tan x \qquad (7)$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) \qquad (8)$$

$$\int \operatorname{cosec}^2 x\,dx = -\cot x \qquad (9)$$

$$\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) \qquad (10)$$

$$\int \sec x\tan x\,dx = \sec x \qquad (11)$$

$$\int \operatorname{cosec} x\cot x\,dx = -\operatorname{cosec} x \qquad (12)$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x = -\cos^{-1} x \tag{13}$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x = -\cot^{-1} x \tag{14}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} = -\cos^{-1} \frac{x}{a} \tag{15}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} = -\frac{1}{a} \cot^{-1} \frac{x}{a} \tag{16}$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x = -\operatorname{cosec}^{-1} x \tag{17}$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} = -\frac{1}{a} \operatorname{cosec}^{-1} \frac{x}{a} \tag{18}$$

$$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \tag{19}$$

* Special case of (1) — when $n = 0$.

$$\int dx = \int 1 \cdot dx = \int x^0\, dx = x. \tag{20}$$

7·4. Illustrative Examples:

1. $\int x^7\, dx = \frac{x^8}{8}$

2. $\int x^{-4}\, dx = -\frac{1}{3} x^{-3}$

3. $\int (3x-2)^3 = \frac{1}{12} (3x-2)^4$

4. $\int \sin(2x+1) = -\frac{1}{2} \cos(2x+1)$

5. $\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1} \frac{x}{3}$

6. $\int \frac{dx}{16+x^2} = \frac{1}{4} \tan^{-1} \frac{x}{4}$

7. $\int \frac{dx}{x\sqrt{x^2-(a+b)^2}} = \frac{1}{a+b} \sec^{-1} \frac{x}{a+b}$

8. $\int \sqrt{5-x^2}\,dx = \frac{x}{2}\sqrt{5-x^2} + \frac{5}{2} \sin^{-1} \frac{x}{\sqrt{5}}$.

7·5. Exercises.

Integrate w. r. t. x

1. x^4
2. x^{-5}
3. $x^{3/2}$
4. $\cos 3x$
5. $\sec 2x \tan 2x$
6. $\operatorname{cosec}^2 (3x+1)$
7. $\sec^2 \frac{x}{2}$
8. $\frac{1}{\sqrt{3-x^2}}$
9. $\frac{1}{4+x^2}$
10. $\frac{-1}{12+x^2}$
11. $\frac{2}{(x-a)^2+(x-b)^2}$
12. $\frac{1}{x\sqrt{x^2-3}}$
13. $\frac{1}{x\sqrt{x^2-\frac{1}{25}}}$
14. $\sqrt{16-x^2}$
15. $\sqrt{7-x^2}$
16. $\frac{5}{(x+1)^2+(2x-1)^2}$.

7·6. Rules of Integration.

Similar to the rules of differentiation we have the following rules for integration.

Rule I. Since the derivative of the sum of a finite number of functions is equal to the sum of the derivatives of the functions, we get that

The integral of the sum of a finite number of functions is equal to the sum of the integrals of the functions.

Symbolically :

$$\int (u+v+w+\cdots)\,dx=\int u\,dx+\int v\,dx+\int w\,dx+\cdots$$

where u, v, w etc. are integrable functions of x.

In particular $\int (u \pm v)\,dx = \int u\,dx \pm \int v\,dx$.

Rule II Since $\frac{d}{dx}[c \cdot \phi(x)] = c \cdot \frac{d}{dx}[\phi(x)]$,

$\therefore$ If $\frac{d}{dx}[\phi(x)] = f(x)$, then we have

$$\frac{d}{dx}[c \cdot \phi(x)] = c\frac{d}{dx}[\phi(x)] = c \cdot f(x).$$

$$\therefore \quad \int c \cdot f(x)\,dx = c \cdot \phi(x)$$

But $\phi(x) = \int f(x)\,dx$.

$$\therefore \quad \int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx.$$

7·7. Illustrative Examples :

1. $\int (x^5+2x^3-x^2+1)\,dx = \frac{1}{6}x^6 + \frac{1}{2}x^4 - \frac{1}{3}x^3 + x.$

2. $\int x(x^2+1)\,dx = \int (x^3+x)\,dx$

$$= \frac{1}{4}x^4 + \frac{1}{2}x^2.$$

3. $\int \sqrt{ax}\,dx = \int \sqrt{a} \cdot \sqrt{x}\,dx$

$$= \sqrt{a}\int x^{\frac{1}{2}}\,dx$$

$$= \sqrt{a} \cdot \frac{2}{3} \cdot x^{3/2}.$$

4. $\int (\sin x + 2\cos x)\, dx = -\cos x + 2\sin x.$

5. $\int \cos^2 dx = \int \left(\frac{1+\cos 2x}{2}\right) dx$

$$= \int \frac{1}{2}\, dx + \int \frac{1}{2} \cos 2x\, dx$$

$$= \frac{1}{2} x + \frac{1}{4} \sin 2x.$$

7·8. Exercises :

Integrate the following :

1. $3x^2 - 4x + 5$

2. $4x^3 - 3x^2 + \frac{7}{x^2}$

3. $(2x+1)^2$

4. $(x+2)^2 + (x+3)^2$

5. $\sec^2 x + \sec x \tan x$

6. $\sqrt{a^2 - x^2} + \frac{1}{\sqrt{a^2 - x^2}}$

7. $\frac{1}{a^2 + x^2} + \frac{1}{b^2 + x^2}$

8. $\sin^2 x.$

7·9. Applications to Mechanics :

We have seen that if s be the distance described by a particle moving along a straight line in time t, then the velocity v at any time t is $\frac{ds}{dt}$ and the acceleration f at time t is $\frac{dv}{dt}$ or $v \frac{dv}{ds}$.

In many problems, involving motion in a straight line, we are given the acceleration. By means of integration, we can determine the velocity and also the distance travelled at any time. Any constant of integration that occurs can be obtained by using the initial conditions of motion.

We consider the following four cases :

(i) acceleration is a constant (uniform acceleration)

(ii) acceleration is a function of distance.

(iii) acceleration is a function of time.

(iv) acceleration is a function of velocity.

* When acceleration is negative it is called *retardation*.

1. A particle starts with initial velocity u, and moves a distance s in time t in a straight line, under constant acceleration f; if v denotes the velocity at time t, we have

$$\frac{dv}{dt} = f.$$

$$\therefore \quad v = \int f\,dt + c.$$ where c is an arbitrary constant.

$$\therefore \quad v = ft + c.$$

When $t = 0,\ v = u.$

$$\therefore \quad u = c$$

$$\therefore \quad \boldsymbol{v = u + ft}.$$

Again $$\frac{ds}{dt} = v.$$

$$= u + ft. \text{ (from above)}$$

$$\therefore \quad s = \int (u + ft)\,dt.$$

$$= ut + \tfrac{1}{2} ft^2 + c'$$ where c' is an arbitrary constant.

But when $t = 0,\ s = 0.$

$$\therefore \quad c' = 0$$

$$\therefore \quad \boldsymbol{s = ut + \tfrac{1}{2} ft^2},$$

Another expression for acceleration is

$$v\frac{dv}{ds} = \tfrac{1}{2}\frac{d}{ds}(v^2).$$

$$\therefore \quad \tfrac{1}{2}\frac{d}{ds}(v^2) = f.$$

$$\therefore \qquad v^2 = \int 2\,f ds.$$

$= 2\,fs + c''$. where c'' is an arbitrary constant.

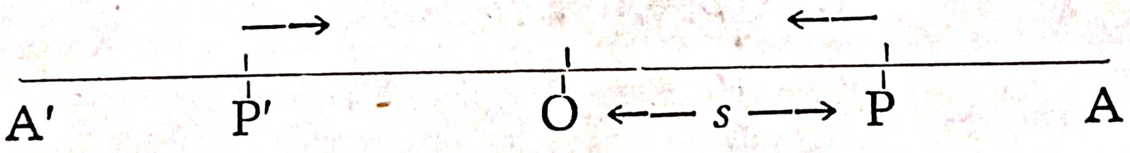
But when $\qquad v = u,\ s = 0.$

$$\therefore \qquad u^2 = c''.$$

$$\therefore \qquad \boldsymbol{v^2 = u^2 + 2\,fs}.$$

* Falling bodies have uniform acceleration due to the force of gravity.

2. A particle starts from rest at a distance a from the origin O and moves with acceleration proportional to its distance from O and directed towards O; find the motion.



Take the form $v\dfrac{dv}{ds} = \tfrac{1}{2}\dfrac{d}{ds}(v^2)$ for the acceleration because in this case the acceleration varies as s.

The acceleration at a distance s from O may be taken as $-\,n^2 s$; for if s is positive, acceleration is negative and is therefore directed towards O. At P$'$, s is negative and therefore $-n^2 s$ is positive which again means that it is directed towards O.

We have $\frac{1}{2}\frac{d}{ds}(v^2) = -\,n^2 s$ where n is some constant.

$$\therefore \quad \frac{1}{2}v^2 = -\,n^2 \int s\,ds.$$

$$= -n^2 \cdot \frac{1}{2}s^2 + c.$$ where c is an arbitrary constant.

But $\quad v = 0$ when $s = a$.

$$\therefore \quad 0 = -\frac{1}{2}n^2\,a^2 + c.$$

$$\therefore \quad c = \frac{1}{2}n^2\,a^2$$

$$\therefore \quad \frac{1}{2}v^2 = -\frac{1}{2}n^2\,s^2 + \frac{1}{2}n^2\,a^2.$$

$$\therefore \quad v^2 = (\,a^2 - s^2\,)$$

$$\therefore \quad v = \pm\, n\sqrt{a^2 - s^2}.$$

It is seen from this equation that $v = 0$ when $s = \pm\, a$.

$\therefore$ If OA′ = OA numerically and OA $= a$, the velocity vanishes at A and A′; also the velocity is maximum when $s = 0$ i. e. at O; and the maximum velocity is na numerically.

Now as the particle is going from A to A′, v is negative and we have

$$\frac{ds}{dt} = v$$

$$= -\,n\sqrt{a^2 - s^2}$$ if n is positive

$$\therefore \quad \frac{dt}{ds} = \frac{1}{\left(\frac{ds}{dt}\right)}$$

$$= -\frac{1}{n\sqrt{a^2 - s^2}}$$

$$\therefore \quad nt = -\int \frac{ds}{\sqrt{a^2 - s^2}}$$

$$= \cos^{-1}\left(\frac{s}{a}\right) + c$$ where c is an arbitrary constant.

But $\quad s = a$ when $t = 0$.

$$\therefore \quad n \cdot 0 = \cos^{-1}\left(\frac{a}{a}\right) + c$$

i. e. $\quad 0 = \cos^{-1} 1 + c$

i. e. $\quad 0 = 0 + c$

$$\therefore \quad c = 0.$$

$$\therefore \quad nt = \cos^{-1}\left(\frac{s}{a}\right)$$

$$\therefore \quad \boldsymbol{s = a \cos nt.}$$

The distance of A′ is $-a$.

So put $s = -a$ in the above equation to get the time t, from A to A′.

$$\therefore \quad -a = a \cos nt_1$$

$$\therefore \quad t_1 = \frac{\pi}{n}.$$

At A′ the velocity is zero just as at A. Its distance from O is also the same as that of A from O i. e. a numerically. The acceleration at A′ is the same as that at A and directed towards O. Therefore the time from A′ to A is the same as that from A to A′ i. e. equal to $\frac{\pi}{n}$.

The motion from A to A′ and back again to A is called *one complete oscillation.*

The time for one complete oscillation is called the *period.*

Thus the period $\mathbf{T} = 2t_1 = \dfrac{2\pi}{n}$.

* Note that the period is independent of a.

The motion described above is called **Simple Harmonic Motion.**

When a particle moves in a straight line so that its acceleration is always directed towards a fixed point in the line and is proportional to the distance from the point, its motion is called *Simple Harmonic Motion.*

The motion of a simple pendulum is simple harmonic.

3. If $at + b$ is the acceleration of a point moving in a straight line, find its velocity (v) after time t, and the space (s) described, given that when $t = 0$; $s = c$ and $v = u$.

Given $\dfrac{dv}{at} = at + b.$

$$\therefore \quad v = \int (at + b)\, dt$$

$$= \int at\, dt + b \int dt$$

$$= \tfrac{1}{2} at^2 + bt + k$$

where k is an arbitrary constant.

But $\quad v = u$ when $t = 0$

$$\therefore \quad u = k.$$

$$\therefore \quad v = \tfrac{1}{2} at^2 + bt + u \qquad \ldots \qquad \ldots \quad (\text{i})$$

which gives the velocity after time t

$$\therefore \quad \frac{ds}{dt} = \tfrac{1}{2} at^2 + bt + u$$

Putting $s = 3000$ ft. in the above equation, we get $t = 1{\cdot}38$.

$\therefore$ The time for traversing 1000 yards is 1·38 secs.

7·10. Illustrative Examples.

1. *Given velocity as a function of time.*

The speed of a moving particle, moving in a straight line is given by $v = 526 - 32t$. How far will the body move from $t = 0$ to $t = 4$?

$$v = 256 - 32t.$$

$$\therefore \quad \frac{ds}{dt} = 256 - 32t.$$

$$\therefore \quad s = \int (256 - 32t)\,dt.$$

$$= 256t - 16t^2 + c$$

where c is an arbitrary constant.

But when $t = 0$, $s = 0$.

$$\therefore \quad c = 0$$

$$\therefore \quad s = 256t - 16t^2$$

Putting $t = 4$, we get $s = 768$

$\therefore$ The particle moves a distance of 768 from $t = 0$ to $t = 4$.

2. *Given velocity as a function of distance.*

Observations made on a rifle-bullet for reasonable distance, give the following relation between the velocity and the distance covered by the bullet

$$v = \frac{1}{2as + b}.$$

where a and b are constants.

It was estimated that the bullet left the muzzle with a velocity of 2400 ft. per sec. and that the velocity was reduced to 2350 ft. per sec. when 100 yards had been traversed. Neglecting gravity, calculate the time of traversing 1000 yards.

We have first to find the values of a and b using the given data

When $s = 0; \quad v = 2400$

When $s = 300; \quad v = 2350.$

∴ We have the two equations

$$2400 = \frac{1}{2a \cdot 0 + b} \quad \cdots \quad \cdots \quad \cdots \quad \text{(i)}$$

$$2350 = \frac{1}{2a \cdot 300 + b} \quad \cdots \quad \cdots \quad \text{(ii)}$$

From (i) we get $b = \dfrac{1}{2400}$

Using this value in (ii) we get $a = \dfrac{1}{12 \times 2400} \times 2350$

Now take $v = \dfrac{1}{2as + b}$

$$\therefore \quad \frac{ds}{dt} = \frac{1}{2as + b}$$

$$\therefore \quad \frac{dt}{ds} = 2as + b$$

$$\therefore \quad t = \int (2as + b)\, ds$$

$= as^2 + bs + c$ where c is an arbitrary constant.

But when $s = 0;\ t = 0.$

$$\therefore \quad c = 0.$$

$$\therefore \quad t = as^2 + bs.$$

Put $s = 3000$ in this relation

we get $t = 1{\cdot}38$ secs.

3. A stone is thrown vertically downwards from the edge of a cliff with a velocity of 20 ft. per sec. Its acceleration is 32 ft./sec^2. Find the velocity with which it hits the water 4 secs. later. Also find the height of the cliff.

Measure the distance from the top of the cliff downwards.

Using $s = ut + \frac{1}{2} ft^2$

and putting $u = 20,\ f = 32;\ t = 4$

we have
$$\begin{aligned} s &= 20 \times 4 + \tfrac{1}{2} \times 32 \times 16 \\ &= 80 + 256 \\ &= 336. \end{aligned}$$

$\therefore$ The height of the cliff = 336 ft.

Now use
$$\begin{aligned} v^2 &= u^2 + 2fs. \\ &= 400 + 2 \times 32 \times 336 \\ &= 400 + 64 \times 336 \\ &= 148^2 \end{aligned}$$

$$\therefore \quad v = 148.$$

$\therefore$ Velocity with which the stone hits the water = 148 ft./sec.

4. A car passes a policeman and continues at the constant speed of 750 yards per min. for one minute. Then its acceleration is $100\ (2 - t)$ yds. per min.^{2}, where t min. is the time since passing the policeman. What is its speed 2 min. after passing the policeman?

The speed is constant, i. e. 750 yds. per min. from $t = 0$ to $t = 1$

$$\frac{dv}{dt} = 100\,(2 - t) \text{ after the first minute}$$

$$\therefore \quad v = \int 100\,(2 - t)\,dt$$

$= 200\,t - 50\,t^2 + c$ where c is an arbitrary constant.

But $v = 750$ when $t = 1$

$\therefore \; 750 = 200 - 50 + c$

$\therefore \quad c = 600$

$\therefore \quad v = 200\,t - 50\,t^2 + 600$

Putting $t = 2$, we get $v = 800$

$\therefore$ The speed of the car after 2 min. is 800 ft./min.

7·11. Exercises:

1. The velocity of a body after t secs. is given by $v = 3\,(t^2 + \frac{1}{4}\,t)$ ft/sec. How far does the body travel in the first four seconds?

2. A body starts from rest and moves so that $v = 24t\,(1 - t)$ ft/sec. after t secs. How far has it travelled when it is again at rest?

3. A body is thrown vertically upwards and after t seconds, its velocity is $(90 - 32t)$ ft/secs. What is its height after (*a*) 2 secs. (*b*) 5 secs.? After how long and with what velocity does it return to its starting-point?

4. The retardation due to gravity of a particle going up an inclined plane is $g \sin \alpha$ where α is the angle of the inclined plane with the horizontal. $g = 32$ ft/sec². A particle shot up an inclined plane of 30° comes to rest at a distance of 2 ft. Show that its initial velocity was 8 ft/sec.

5. A stone is sent gliding over ice with initial velocity of 40 ft/sec. If friction leads to a retardation of $\frac{16}{5}$ ft/sec.^{2} show that the stone will go 250 ft. before it stops.

6. A body starts from rest at $t = 0$ and moves with an acceleration $(t + 1)$ ft/sec^2., where t secs. is the time from the start. What is the velocity after 4 secs? What is the distance travelled after 3 secs.?

7. A body starts with velocity $\frac{2}{3}$ ft/sec and after t secs. its acceleration is $(1 - t)$ ft/sec^2. After how long will it be back at the starting point? What will be its velocity then?

8. A train starts from rest at a station and after t min. its acceleration is $900t(2 - t)$ yds per min.^{2} until this expression vanishes, when the acceleration remains zero until the train puts the brakes for the next station. Find the maximum velocity of the train between stations and the distance in which this velocity is attained.

9. A particle is moving according to the law $v = v_0 \cos nt$; show that the space described by the time it first comes to rest is $\frac{v_0}{n}$. (Lamb)

Answers.

Ex. 7·5.

1. $\frac{x^5}{5}$
2. $-\frac{x^{-4}}{4}$
3. $\frac{2}{5}x^{5/2}$
4. $\frac{\sin 3x}{3}$
5. $\frac{\sec 2x}{2}$
6. $\frac{-\cot(3x+1)}{3}$
7. $2\tan\frac{x}{2}$
8. $\sin^{-1}\left(\frac{x}{\sqrt{3}}\right)$
9. $\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$

10. $\frac{1}{2\sqrt{3}}\cot^{-1}\left(\frac{x}{2\sqrt{3}}\right)$

11. $\frac{2}{\sqrt{3a^2-2ab+3b^2}}\tan^{-1}\left[\frac{2x+a+b}{\sqrt{3a^2-2ab+3b^2}}\right]$

12. $\frac{1}{\sqrt{3}}\sec^{-1}\left(\frac{x}{\sqrt{3}}\right)$

13. $5\sec^{-1}(5x)$

14. $\frac{x}{2}\sqrt{16-x^2}+8\sin^{-1}\left(\frac{x}{4}\right)$

15. $\frac{x}{2}\sqrt{7-x^2}+\frac{7}{2}\sin^{-1}\left(\frac{x}{\sqrt{7}}\right)$

16. $\frac{10}{\sqrt{31}}\tan^{-1}\left(\frac{10x-3}{\sqrt{31}}\right)$.

Ex. 7·8.

1. x^3-2x^2+5x
2. $x^4-x^3-\frac{7}{x}$
3. $\frac{4}{3}x^3+2x^2+x$
4. $\frac{2}{3}x^3+5x^2+13x$.
5. $\tan x+\sec x$
6. $\frac{x}{2}\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}$
7. $\frac{1}{a}\tan^{-1}\frac{x}{a}+\frac{1}{b}\tan^{-1}\frac{x}{b}$
8. $\frac{1}{2}x-\frac{1}{4}\sin 2x$.

Ex. 7·11.

1. 75 ft.
2. 4 ft.
3. (a) 116 ft. (b) 50 ft.; $5\frac{5}{8}$ secs., -90 ft./sec.
6. 12 ft./sec.; 9 ft.
7. 4 secs.; $-3\frac{1}{3}$ ft./sec.
8. 1200 yd./min. (= 41 m. p. h. nearly); 1200 yds.

CHAPTER VIII

METHODS OF INTEGRATION

8·1. We have defined integration as the inverse process of differentiation in the last chapter. Given simple functions as derivatives, we have seen how to find their primitives i. e. functions of which the given functions are the derivatives.

In the present chapter we shall give two powerful methods of integration viz :

(i) The method of substitution

and (ii) The method of integration by parts.

With the help of these methods we can integrate many of the functions the primitives for which cannot be obtained directly by inspection.

8·2. Change of the independent variable.

(or the method of substitution.)

The method of substitution in differentiation of a function was explained in § 4·5. Given y, a function of x, where x itself is a function of t, the derivative of y w. r. to t was given by the formula

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

By definition of the integral, we have

$$y = \int \frac{dy}{dt} \cdot dt$$

$$= \int \frac{dy}{dx} \cdot \frac{dx}{dt} \cdot dt \qquad \ldots \qquad \ldots \qquad (A)$$

It may happen that by using a suitable substitution for x, in terms of another variable t, the integral $\int \frac{dy}{dx} \cdot \frac{dx}{dt} \cdot dt$ is simpler to evaluate than $\int \frac{dy}{dx}\, dx$.

An example will illustrate how this is effected.

Consider $\frac{dy}{dx} = \cos^3 x$

i. e. $y = \int \cos^3 x\, dx.$

Now if we write $\sin x = t$, then

$$\cos x \cdot \frac{dx}{dt} = 1$$

$$\therefore \quad \frac{dx}{dt} = \frac{1}{\cos x}$$

$$\begin{aligned} \therefore \quad \int \cos^3 x\, dx &= \int \cos^3 x \frac{dx}{dt}\, dt \\ &= \int \cos^3 x \cdot \frac{1}{\cos x}\, dt \\ &= \int (1 - \sin^2 x)\, dt \\ &= \int (1 - t^2)\, dt \qquad \ldots \qquad \ldots \qquad (B) \\ &= t - \tfrac{1}{3} t^3 \\ &= \sin x - \tfrac{1}{3} \sin^3 x. \end{aligned}$$

Thus the integral was brought to the simple form (B) which can be evaluated directly.

The example solved above shows that the integrand in (A) has to be expressed in terms of t by means of the assumed relation between x and t. This can be seen from

(B). The evaluation of the integral is thus obtained in terms of t, which has to be transformed back to an expression in x.

The above has been an example of a *trigonometric substitution.* The other kind is the *algebraic substitution.* We give below an example to illustrate the latter also.

Take $\int x \sin x^2 \, dx.$

Put $x^2 = t$ so that $\dfrac{dx}{dt} = \dfrac{1}{2x}$

$$\therefore \int x \sin x^2 \, dx = \int x \sin x^2 \frac{dx}{dt} \, dt.$$

$$= \int x \sin x^2 \cdot \frac{1}{2x} \, dt.$$

$$= \frac{1}{2} \int \sin x^2 \, dt.$$

$$= \frac{1}{2} \int \sin t \, dt.$$

$$= -\frac{1}{2} \cos t.$$

$$= -\frac{1}{2} \cos x^2.$$

To prove that $\int f(x) \, dx = \int f(x) \cdot \dfrac{dx}{dt} \, dt$ *where* x *is some function of* t.

Proof :—Let $y = \int f(x) \, dx.$

$$\therefore \frac{dy}{dx} = f(x).$$

But x is a function of t.

$$\therefore \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

$$= f(x)\frac{dx}{dt}.$$

$$\therefore y = \int f(x)\frac{dx}{dt}\,dt.$$

$$\therefore \int f(x)\,dx = \int f(x)\frac{dx}{dt}\,dt.$$

8·3 Illustrative Examples :—

1. Integrate $(3x+4)^{3/2}$ w. r. t. x.

Put $z = (3x+4)$

$\therefore 1 = 3\frac{dx}{dz}$ (differentiating w. r. t. z.)

$$\therefore \frac{dx}{dz} = \frac{1}{3}$$

$$\text{Now} \int (3x+4)^{3/2}\,dx = \int (3x+4)^{3/2}\frac{dx}{dz}\cdot dz$$

$$= \int z^{3/2} \cdot \left(\frac{1}{3}\right) dz$$

$$= \frac{1}{3}\int z^{3/2}\,dz$$

$$= \frac{1}{3}\cdot\frac{2}{5}\,z^{5/2}$$

$$= \frac{2}{15}(3x+4)^{5/2}$$

2. $\displaystyle\int \frac{x\,dx}{\sqrt{a^4-x^4}}$.

Method I. Put $x^2 = u$ so that $2x\dfrac{dx}{du} = 1$

$$\therefore \quad \frac{dx}{du} = \frac{1}{2x}$$

$$\therefore \quad \int \frac{x\,dx}{\sqrt{a^4-x^4}} = \int \frac{x}{\sqrt{a^4-x^4}} \cdot \frac{dx}{du} \cdot du$$

$$= \int \frac{x}{\sqrt{a^4-x^4}} \cdot \frac{1}{2x} \cdot du$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{a^4-u^2}}\, du$$

$$= \frac{1}{2} \sin^{-1}\left(\frac{u}{a^2}\right)$$

$$= \frac{1}{2} \sin^{-1}\left(\frac{x^2}{a^2}\right)$$

Method II. Put $x^2 = a^2 \sin\theta$ so that $\theta = \sin^{-1}\dfrac{x^2}{a^2}$

$$\therefore \quad 2x\frac{dx}{d\theta} = a^2\cos\theta$$

$$\therefore \quad \frac{dx}{d\theta} = \frac{a^2}{2x}\cos\theta$$

$$\therefore \quad \int \frac{x}{\sqrt{a^4-x^4}}\, dx = \int \frac{x}{\sqrt{a^4-x^4}}\, \frac{dx}{d\theta} \cdot d\theta$$

$$= \int \frac{x}{\sqrt{a^4-x^4}} \cdot \frac{a^2\cos\theta}{2x}\, d\theta$$

$$= \int \frac{a^2\cos\theta}{2\sqrt{a^4-a^4\sin^2\theta}}\, d\theta.$$

$$= \int \frac{a^2\cos\theta}{2a^2\sqrt{1-\sin^2\theta}}\, d\theta.$$

$$= \frac{1}{2}\int \frac{\cos\theta\, d\theta}{\cos\theta}.$$

$$= \frac{1}{2}\int d\theta.$$

$$= \frac{1}{2}\theta$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{x^2}{a^2}\right).$$

3. $\displaystyle\int \frac{dx}{(a^2+x^2)^{3/2}}.$

Method I. Put $x = \dfrac{1}{t}$

$$\therefore \quad \frac{dx}{dt} = -\frac{1}{t^2}$$

$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \int \frac{1}{\left(a^2+\frac{1}{t^2}\right)^{3/2}}\,\frac{dx}{dt}\cdot dt.$$

$$= \int \frac{t^3}{(a^2t^2+1)^{3/2}}\cdot\left(\frac{-1}{t^2}\right) dt$$

$$= \int \frac{-t\, dt}{(a^2t^2+1)^{3/2}}$$

$$= -\frac{1}{2a^2}\int \frac{2a^2t\, dt}{(a^2t^2+1)^{3/2}}$$

$$= -\frac{1}{2a^2}\int \frac{du}{(u+1)^{3/2}} \text{ where } u = a^2t^2$$

$$= -\frac{1}{2a^2}\int (u+1)^{-3/2}\, du$$

$$= -\frac{1}{2a^2}\cdot\frac{-2}{1}(u+1)^{-1/2}$$

$$= \frac{1}{a^2} \frac{1}{\sqrt{a^2t^2+1}}$$

$$= \frac{1}{a^2} \cdot \frac{x}{\sqrt{a^2+x^2}} \text{ putting } t = \frac{1}{x}.$$

II Method Put $x = a \tan \theta$

$$\therefore \quad \frac{dx}{d\theta} = a \sec^2 \theta$$

$$\therefore \quad \int \frac{dx}{(a^2+x^2)^{3/2}} = \int \frac{1}{(a^2+a^2 \tan^2\theta)^{3/2}} \frac{dx}{d\theta} \cdot d\theta$$

$$= \int \frac{1}{(a^2 \sec^2 \theta)^{3/2}} \times \frac{a \sec^2 \theta}{1} d\theta$$

$$= \int \frac{a \sec^2 \theta}{a^3 \sec^3 \theta} d\theta$$

$$= \frac{1}{a^2} \int \cos \theta \, d\theta$$

$$= \frac{1}{a^2} \sin \theta.$$

But $\tan \theta = \frac{x}{a}$

$$\therefore \quad \sin \theta = \frac{x}{\sqrt{a^2+x^2}}$$

$$\therefore \quad \int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{1}{a^2} \cdot \frac{x}{\sqrt{a^2+x_2}}$$

Note :—The examples 2. and 3. worked above show that of the two kinds of substitutions, viz. algebraic or trigonometric, in an integral, one of the two substitutions, *mostly the trigonometric one*, simplifies the process of integration better than the other.

The student is recommended to use both these methods wherever possible, for only then will he begin to see which substitution is more powerful in any particular case.

4. $\int \frac{1}{x^4}\sqrt{1+x^2}\,dx.$

Put $x=\frac{1}{t}$ so that $\frac{dx}{dt}=-\frac{1}{t^2}$

$$\therefore \int \frac{1}{x^4}\sqrt{1+x^2}\,dx = \int \frac{1}{x^4}\sqrt{1+x^2}\,\frac{dx}{dt}\,dt$$

$$= \int t^4\sqrt{1+\frac{1}{t^2}}\cdot\left(\frac{-1}{t^2}\right)dt$$

$$= \int \frac{-t^4\sqrt{t^2+1}}{t^3}\,dt$$

$$= -\int \sqrt{1+t^2}\cdot t\,dt$$

$$= \frac{-1}{2}\int \sqrt{1+t^2}\cdot 2t\,dt$$

$$= \frac{-1}{2}\int \sqrt{1+u}\,du \text{ where } u=t^2$$

$$= \frac{-1}{2}\cdot\frac{2}{3}(1+u)^{3/2}$$

$$= \frac{-1}{3}(1+t^2)^{3/2}$$

$$= \frac{-1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}$$

$$= \frac{-1}{3}\cdot\frac{(x^2+1)^{3/2}}{x^3}.$$

5. $\int \frac{1}{5+3\cos x}\,dx$

Put $t = \tan\frac{x}{2}$

$$\therefore \quad \cos x = \frac{1-t^2}{1+t^2}$$

Also differentiating w. r. t. t

$$1 = \sec^2\frac{x}{2} \times \frac{1}{2} \cdot \frac{dx}{dt}$$

$$\therefore \quad \frac{dx}{dt} = \frac{2}{\sec^2\frac{x}{2}} = \frac{2}{1+\tan^2\frac{x}{2}}$$

$$= \frac{2}{1+t^2}$$

$$\therefore \quad \int \frac{1}{5+3\cos x} = \int \frac{1}{5+\frac{3(1-t^2)}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2dt}{5(1+t^2)+3(1-t^2)}$$

$$= \int \frac{2dt}{8+2t^2}$$

$$= \int \frac{dt}{4+t^2}$$

$$= \frac{1}{2}\tan^{-1}\frac{1}{2}t$$

$$= \frac{1}{2}\tan^{-1}\left[\frac{1}{2}\tan\frac{x}{2}\right]$$

6. $\int \sqrt{a^2-x^2}\,dx$

Put $x = a\sin\theta$.

$$\therefore \quad \frac{dx}{d\theta} = a\cos\theta.$$

$$\therefore \int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - x^2}\, \frac{dx}{d\theta} \cdot d\theta$$

$$= \int \sqrt{a^2 - a^2 \sin^2 \theta} \cdot a \cos \theta \cdot d\theta$$

$$= \int a \cos \theta \cdot a \cos \theta \cdot d\theta$$

$$= a^2 \int \cos^2 \theta \, d\theta$$

$$= \frac{a^2}{2} \int (\cos 2\theta + 1)\, d\theta$$

$$= \frac{a^2}{2} \left(\tfrac{1}{2} \sin 2\theta + \theta \right)$$

$$= \frac{a^2}{2} \sin \theta \cos \theta + \frac{a^2}{2} \theta$$

$$= \frac{a^2}{2} \cdot \frac{x}{a} \cdot \sqrt{1 - \frac{x^2}{a^2}} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right)$$

$$= \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right)$$

7. $\displaystyle\int \frac{1}{1 + x^2} \cos^{-1} \left(\frac{1 - x^2}{1 + x^2} \right) dx.$

Put $x = \tan \theta$ so that $\dfrac{dx}{d\theta} = \sec^2\theta$

$$= 1 + \tan^2\theta$$

$$= 1 + x^2$$

$$\therefore \int \frac{1}{1+x^2} \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx = \int \frac{1}{1+x^2} \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \frac{dx}{d\theta}\, d\theta$$

$$= \int \frac{1}{1+x^2} \cos^{-1}\left[\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right] \times (1+x^2)\, d\theta.$$

$$= \int \cos^{-1}\left[\cos 2\theta\right] d\theta$$

$$= \int 2\theta \, d\theta$$

$$= \theta^2$$

$$= (\tan^{-1} x)^2.$$

8·4. Integration by Parts.

We know that if u and v are differentiable functions of x, then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\therefore \quad uv = \int \left(u\frac{dv}{dx} + v\frac{du}{dx}\right) dx.$$

$$= \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}\,dx.$$

$$\therefore \quad \int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$

This is called the *rule of integration by parts*. It is used when the integral on the right-hand side is simpler than the integral on the left-hand side.

* Note a particular case of this when $v = x$ so that

$$\frac{dv}{dx} = 1.$$

Then $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$

becomes

$$\int u.\,1.\,dx = u \cdot x - \int x\frac{du}{dx}\,dx.$$

∴ We have the particular case

$$\int u\,dx = ux - \int x\,\frac{du}{dx}\,dx.$$

8·5. Illustrative Examples :

1. $\int \sqrt{a^2-x^2}\,dx$

Here $u=\sqrt{a^2-x^2}$ and $v=x$.

$$\therefore \quad \int\sqrt{a^2-x^2}\,dx = \sqrt{a^2-x^2}\cdot x - \int x\cdot\frac{1}{2}\cdot\frac{-2x\,dx}{\sqrt{a^2-x^2}}$$

$$= x\sqrt{a^2-x^2} + \int\frac{x^2\,dx}{\sqrt{a^2-x^2}} \quad \cdots(1)$$

But

$$\int\sqrt{a^2-x^2} = \int\frac{a^2-x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= a^2\int\frac{dx}{\sqrt{a^2-x^2}} - \int\frac{x^2\,dx}{\sqrt{a^2-x^2}}$$

$$= a^2\sin^{-1}\left(\frac{x}{a}\right) - \int\frac{x^2\,dx}{\sqrt{a^2-x^2}} \quad \cdots(2)$$

Adding (1) and (2) we get

$$2\int\sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \quad \int\sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right).$$

Note :—The above example was also worked by the method of trigonometric substitution in Ex. 6 of § 8·2. Here it is worked again by the method of integration by parts. A comparison of the two methods for the same example will show that sometimes integration by parts is more direct and more simple than integration by substitution.

2. $$\int x \cos x \, dx = \int x \frac{d}{dx}(\sin x)\, dx$$
$$= x \sin x - \int \sin x \cdot \frac{d}{dx}(x)\, dx$$
$$= x \sin x - \int \sin x \, dx$$
$$= x \sin x + \cos x$$

3. $$\int x \sin x \cos x \, dx = \frac{1}{2}\int x \sin 2x \, dx$$
$$= \frac{1}{2}\int x \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right) dx$$
$$= -\frac{1}{4} x \cos 2x - \frac{1}{2}\int\left(-\frac{1}{2}\cos 2x\right)\frac{d}{dx}(x)dx$$
$$= -\frac{1}{4} x \cos 2x + \frac{1}{4}\int \cos 2x \, dx$$
$$= -\frac{1}{4} x \cos 2x + \frac{1}{8}\sin 2x.$$

4. $$\int \sin^{-1} dx \cdot = \int \sin^{-1} x \cdot \frac{d}{dx}(x)\, dx$$
$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\, dx$$
$$= x \sin^{-1} x - \frac{1}{2}\int \frac{2x \, dx}{\sqrt{1-x^2}}$$
$$= x \sin^{-1} x - \frac{1}{2}\int \frac{du}{\sqrt{1-u}}$$
where $u = x^2$
$$= x \sin^{-1} x + \sqrt{1-u}$$
$$= x \sin^{-1} x + \sqrt{1-x^2}.$$

2. $$\int x \cos x \, dx = \int x \frac{d}{dx}(\sin x)\, dx$$

$$= x \sin x - \int \sin x \cdot \frac{d}{dx}(x)\, dx$$

$$= x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x$$

3. $$\int x \sin x \cos x \, dx = \frac{1}{2}\int x \sin 2x \, dx$$

$$= \frac{1}{2}\int x \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right) dx$$

$$= -\frac{1}{4}x \cos 2x -$$

$$\frac{1}{2}\int\left(-\frac{1}{2}\cos 2x\right)\frac{d}{dx}(x) dx$$

$$= -\frac{1}{4}x\cos 2x + \frac{1}{4}\int \cos 2x \, dx$$

$$= -\frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x.$$

4. $$\int \sin^{-1} dx \cdot = \int \sin^{-1} x \cdot \frac{d}{dx}(x)\, dx$$

$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\, dx$$

$$= x \sin^{-1} x - \frac{1}{2}\int \frac{2x\, dx}{\sqrt{1-x^2}}$$

$$= x \sin^{-1} x - \frac{1}{2}\int \frac{du}{\sqrt{1-u}}$$

where $u = x^2$

$$= x \sin^{-1} x + \sqrt{1-u}$$

$$= x \sin^{-1} x + \sqrt{1-x^2}.$$

5. $$\int \tan^4 x\,dx = \int \tan^2 x\,(\sec^2 x - 1)\,dx$$

$$= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx$$

$$= \int \tan^2 x \sec^2 x\,dx - \int (\sec^2 x - 1)\,dx$$

$$= \int \tan^2 x \sec^2 x\,dx - \int \sec^2 x\,dx + \int dx$$

$$= \int u^2\,du - \int du + \int dx \quad \text{where}$$

$$u = \tan x$$

$$= \frac{u^3}{3} - u + x$$

$$= \frac{1}{3}\tan^3 x - \tan x + x.$$

6. $$\int \frac{x + \sin x}{1 + \cos x}\,dx = \int \frac{x\,dx}{1 + \cos x} + \int \frac{\sin x}{1 + \cos x}\,dx$$

$$= \int \frac{x\,dx}{2\cos^2 \frac{x}{2}} + \int \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\,dx$$

$$= \frac{1}{2}\int x\sec^2\frac{x}{2}\,dx + \int \tan\frac{x}{2}\,dx$$

$$= \frac{1}{2}\int x\sec^2\frac{x}{2}\,dx + x\tan\frac{x}{2}$$

$$- \frac{1}{2}\int x\sec^2\frac{x}{2}\,dx$$

$$= x\tan\frac{x}{2}.$$

8·6. Integration by trigonometric transformation :

The following trigonometric formulæ are useful in integration.

Formulæ :

(i) $\sin^2 x + \cos^2 x = 1$

$\therefore \quad \sin^2 x = 1 - \cos^2 x$

and $\quad \cos^2 x = 1 - \sin^2 x$

(ii) $\tan^2 x = 1 - \sec^2 x$

(iii) $\sin 2x = 2 \sin x \cos x$

$\therefore \quad \sin x \cos x = \frac{1}{2} \sin 2x$

(iv) $\cos 2x = 1 - 2 \sin^2 x$

$= 2 \cos^2 x - 1$

$\therefore \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$

and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

(v) $\sin 3x = 3 \sin x - 4 \sin^3 x$

$\therefore \quad \sin^3 x = \frac{1}{4}(3 \sin x - \sin 3x)$

(vi) $\cos 3x = 4 \cos^3 x - 3 \cos x$

$\therefore \quad \cos^3 x = \frac{1}{4}(3 \cos x + \cos 3x)$

(vii) $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

(viii) $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

(ix) $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

(x) $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

The following examples illustrate the use of some of the above formulae.

1. $$\int \sin^3 x\, dx = \frac{1}{4}\int (3 \sin x - \sin 3x)\, dx$$
$$= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x$$

2. $$\int \tan^2 x dx = \int (\sec^2 x - 1)\, dx$$
$$= \int \sec^2 x dx - \int 1 \cdot dx$$
$$= \tan x - x$$

3. $$\int \sin x \cos x dx = \frac{1}{2}\int \sin 2\, x dx$$
$$= -\frac{1}{4}\cos 2\, x$$

4. $$\int \cos^2 x dx = \frac{1}{2}\int (1 + \cos 2x)\, dx$$
$$= \frac{1}{2}x + \frac{1}{4}\sin 2x$$

5. $$\int \sin 5x \cos 3x dx = \frac{1}{2}\int 2 \sin 5x \cos 3x dx$$
$$= \frac{1}{2}\int \Big[\sin 8x + \sin 2x\Big]\, dx$$
$$= \frac{1}{2}\left(-\frac{1}{8}\right)\cos 8x + \frac{1}{2}\left(-\frac{1}{2}\right)\cos 2x$$
$$= -\left(\frac{1}{16}\cos + \frac{1}{4}\cos 2x\right).$$

6. $$\int \sin 3x \sin 4\, x dx = \frac{1}{2}\int 2 \sin 3x \sin 4x\, dx$$
$$= \frac{1}{2}\int \Big[\cos x - \cos 7x\Big] dx$$

$$= \frac{1}{2} \sin x - \frac{1}{14} \sin 7x$$

8·7 Miscellaneous Examples

1. $$\int \frac{x\,dx}{\sqrt{a+x}} = \int \frac{(x+a) - a}{\sqrt{a+x}}\, dx$$

$$= \int (a+x)^{1/2}\, dx - a \int \frac{dx}{(a+x)^{1/2}}$$

$$= \frac{2}{3} (x + a)^{3/2} - 2a\,(x + a)^{1/2}$$

2. $$\int \frac{\sqrt{x^2 - a^2}}{x}\, dx$$

Put $x = a \sec \theta$.

$$\therefore \quad \frac{dx}{d\theta} = a \sec \theta \tan \theta.$$

$$\therefore \quad \int \frac{\sqrt{x^2 - a^2}}{x}\, dx = \int \frac{a \tan \theta}{a \sec \theta} \cdot a \sec \theta \tan \theta\, d\theta$$

$$= a \int \tan^2 \theta\, d\theta.$$

$$= a\,(\tan \theta - \theta) \text{ from Ex. 2 of § 8·6}$$

But $x = a \sec \theta$

$$\therefore \quad \frac{a}{x} = \cos \theta$$

$$\therefore \quad \theta = \cos^{-1} \frac{a}{x}$$

Also $\sec^2 \theta = \frac{x^2}{a^2}$

$$\therefore \quad \tan^2 \theta = \sec^2 \theta - 1$$

$$= \frac{x^2}{a^2} - 1$$

$$= \frac{x^2 - a^2}{a^2}$$

$$\therefore \quad \tan\theta = \frac{\sqrt{x^2-a^2}}{a}$$

$$\therefore \quad \int \frac{\sqrt{x^2-a^2}}{x}\,dx = a\left[\frac{\sqrt{x^2-a^2}}{a} - \cos^{-1}\frac{a}{x}\right]$$

$$= \sqrt{x^2-a^2} - a\cos^{-1}\frac{a}{x}$$

3. $$\int \sqrt{(x+1)(2-x)}\,dx = \int \sqrt{2+x-x^2}\,dx$$

$$= \int \sqrt{\frac{9}{4} - \left(x-\frac{1}{2}\right)^2}\,dx$$

Put $x - \frac{1}{2} = \frac{3}{2}\cos y$

$$\therefore \quad \frac{dx}{dy} = -\frac{3}{2}\sin y.$$

$$\therefore \int \sqrt{(x+1)(2-x)}\,dx = \int \sqrt{(x+1)(2-x)}\,\frac{dx}{dy}\,dy$$

$$= \int \sqrt{\frac{9}{4} - \frac{9}{4}\cos^2 y}\left(-\frac{3}{2}\sin y\right)dy$$

$$= -\frac{9}{4}\int \sin^2 y\,dy.$$

$$= \frac{9}{8}\int -2\sin^2 y\,dy$$

$$= \frac{9}{8}\int \left(\cos 2y - 1\right)dy$$

$$= \frac{9}{8}\left[\tfrac{1}{2}\sin 2y - y\right]$$

$$= \frac{9}{8}\left[\sin y\cos y - y\right]$$

But $\frac{2x-1}{2}=\frac{3}{2}\cos y$

$$\therefore \quad \cos y=\frac{2x-1}{3}$$

$$\therefore \quad \sin y=\frac{2}{3}\sqrt{(x+1)(2-x)}$$

Also $\quad y=\cos^{-1}\left[\frac{2x-1}{3}\right]$

$$\therefore \int\sqrt{(x+1)(2-x)}=\frac{2x-1}{4}\sqrt{(x+1)(2-x)}-\frac{9}{8}\cos^{-1}\left(\frac{2x-1}{3}\right).$$

4. $\int\frac{x\,dx}{\sqrt{5x+4}}$

Put $\sqrt{5x+4}=u$

$$\therefore \quad 5x+4=u^2$$

$$\therefore \quad 5\cdot\frac{dx}{du}=2u.$$

$$\therefore \int\frac{x dx}{\sqrt{5x+4}}=\int\frac{x\,dx}{\sqrt{5x+4}}\cdot\frac{dx}{du}\cdot du$$

$$=\int\frac{u^2-4}{5u}\left(\frac{2u}{5}\right)du.$$

$$=\frac{2}{25}\int(u^2-4)\,du.$$

$$=\frac{2}{75}u^3-\frac{8}{5}u$$

$$=\frac{2}{75}(5x+4)^{3/2}-\frac{8}{25}(5x+4)^{1/2}.$$

$$=\frac{2}{75}(5x+4)^{1/2}(5x+4-12).$$

$$=\frac{2}{75}\sqrt{5x+4}\,(5x-8).$$

8·8. Exercises:

Prove the following:

1. $\displaystyle\int (3x-4)^7\,dx=\frac{(3x-4)^8}{24}$

2. $\displaystyle\int 2x\,(x^2-3)^4\,dx=\frac{(x^2-3)^5}{5}$

3. $\displaystyle\int (1-2x)^{\frac{1}{2}}dx=\frac{-(1-2x)^{3/2}}{3}$

4. $\displaystyle\int x^{n-1}\,(x^n+a^n)^3\,dx=\frac{(x^n+a^n)^4}{4n}$

5. $\displaystyle\int \frac{dx}{\sqrt{1-2x}}=-\sqrt{1-2x}$

6. $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx=-\sqrt{1-x^2}$

7. $\displaystyle\int \frac{2x-5}{\sqrt{x^2-5x+3}}\,dx=2\sqrt{x^2-5x+3}$

8. $\displaystyle\int \frac{x}{(x^2-1)^{3/2}}\,dx=\frac{-1}{\sqrt{x^2-1}}$

9. $\displaystyle\int \frac{\cos x}{4+\sin^2 x}=\frac{1}{2}\tan^{-1}\left[\frac{1}{2}\sin x\right]$

10. $\displaystyle\int \sin^2\frac{x}{2}\,dx=\frac{1}{2}\,(x-\sin x)$

11. $\displaystyle\int \frac{\sec^2 x\,dx}{4+\tan^2 x}=\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan x\right)$

12. $\displaystyle\int \frac{x^2}{\sqrt{x+1}}\,dx=\frac{2}{15}\sqrt{x-1}\,(3x^2-4x+8)$

13. $\displaystyle\int \frac{x\,dx}{\sqrt{x+2}}=\frac{2}{3}\sqrt{x+2}\,(x-4)$

14. $\int \frac{dx}{x^2(x^2+1)} = -\tan^{-1} x - \frac{1}{x}$

15. $\int \frac{\cos x\, dx}{(1-\sin x)^2} = \frac{1}{1-\sin x}$

16. $\int \frac{x dx}{1+x^4} = \frac{1}{2} \tan^{-1} x^2$

17. $\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{1}{a^2} \cdot \frac{x}{\sqrt{a^2-x^2}}$

18. $\int \frac{dx}{(x^2-a^2)^{3/2}} = -\frac{1}{a^2} \cdot \frac{x}{\sqrt{x^2-a^2}}$

19. $\int \frac{dx}{\sqrt{1+x}+\sqrt{x}} = \frac{2}{3}(1+x)^{3/2} - \frac{2}{3}x^{3/2}$

20. $\int \frac{dx}{(1-x)\sqrt{1-x^2}} = \sqrt{\frac{1+x}{1-x}}$

21. $\int \frac{dx}{5+4\cos x} = \frac{2}{3} \tan^{-1}\left(\frac{1}{3} \tan \frac{x}{2}\right)$

22. $\int \frac{dx}{(1+x^2)^2} = \frac{1}{2} \tan^{-1} x + \frac{1}{2} \cdot \frac{x}{1+x^2}$

23. $\int x \sin ax\, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax$

24. $\int \cos^3 x\, dx = \frac{3}{4} \sin x + \frac{1}{2} \sin 3x$

25. $\int \frac{\sqrt{x^2-9}}{x} dx = \sqrt{x^2-9} - 3\cos^{-1}\left(\frac{3}{x}\right)$

26. $\int \sec^6 x\, dx = \tan x + \frac{2}{3} \tan^3 x + \frac{1}{5} \tan^5 x$

27. $\int x^2 \sin 2x\, dx = \frac{1}{4} \cos 2x + \frac{x}{2} \sin 2x - \frac{x^2}{2} \cos 2x$

28. $\int x \tan^{-1} dx = \frac{1}{2}\left[(1+x^2)\tan^{-1} x - x\right]$

29. $\int \sqrt{(x-3)(7-x)}\,dx = \frac{x-5}{2}\sqrt{(x-3)(7-x)} + 4\sin^{-1}\left(\frac{x-3}{2}\right)$

30. $\int \frac{dx}{25-24\sin^2 x} = \frac{1}{5}\tan^{-1}\left(\frac{1}{5}\tan x\right)$

31. $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}} = \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{x\sqrt{2}}{\sqrt{1-x^2}}\right]$

32. $\int \sqrt{\frac{a+x}{a-x}}\,dx = a\sin^{-1}\frac{x}{2} - \sqrt{a^2-x^2}$

33. $\int \frac{dx}{x\sqrt{2ax-x^2}} = \frac{-\sqrt{2ax-x^2}}{ax}$

34. $\int \frac{dx}{\sqrt{2-x-x^2}} = \sin^{-1}\frac{2x+1}{3}$

35. $\int \frac{x^3}{\sqrt{a^2+x^2}}\,dx = \frac{1}{3}(x^2-2a^2)\sqrt{a^2+x^2}$

36. $\int \frac{dx}{x^2\sqrt{2+x^2}} = -\frac{1}{2}\sqrt{\frac{2+x^2}{x^2}}$

37. $\int \frac{dx}{41+9\cos x} = \frac{1}{20}\tan^{-1}\left(\frac{4}{5}\tan\frac{1}{2}x\right)$

38. $\int \frac{dx}{x\sqrt{x^2+2x-3}} = -\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{3-x}{2x}\right)$

39. $\int \sqrt{1+\sin x}\,dx = \frac{1}{2}\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)$

$\left(\text{Hint. } \sqrt{1+\sin x} = \sin\frac{x}{2} + \cos\frac{x}{2}\right)$

40. $\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{1}{a^2}\cdot\frac{x}{\sqrt{a^2-x^2}}$

41. $\int \frac{x^2\,dx}{1+x^6} = \frac{1}{3}\tan^{-1}x^3$

42. $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx = \frac{1}{2} (\sin^{-1}x)^2$

43. $\int \sin x \cos^3 x \, dx = -\frac{1}{4} \cos^4 x.$

44. $\int \frac{dx}{1+\cos x} = \operatorname{cosec} x - \cot x$

45. $\int \frac{dx}{1-\cos x} = -\cot x - \operatorname{cosec} x$

46. $\int \frac{dx}{1+\cos^2 x} = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{1}{\sqrt{2}} \tan x \right)$

47. $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} = -\sqrt{1-x^2} \sin^{-1} x + x$

48. $\int \frac{x\, dx}{\sqrt{a^4-x^4}} = \frac{1}{2} \sin^{-1} \frac{x^2}{a^2}$

49. $\int \frac{dx}{x\sqrt{x^2-a^2}} = -\frac{1}{a} \sin^{-1} \frac{a}{x}$.

50. Integrate $\frac{x^3}{(x^2+1)^3}$

(i) by the substitution $u = x^2 + 1$

(ii) by the substitution $x = \tan\theta$

Explain the difference between the results.